Operations Research
Chapter 3: Existence and Optimality Conditions for Minimizers

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Outline

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Conditions for Local Minimizers

Theorem 1 (**Second-Order Necessary Condition (SONC), Boundary Case**)

Let $f \in \mathcal{C}^2(\boldsymbol{\Omega})$ be a real-valued function defined on some set $\boldsymbol{\Omega} \subset \mathbb{R}^n$, whose boundary is denoted by $\boldsymbol{\partial\Omega}$. Let also $\boldsymbol{x}^* \in \boldsymbol{\partial\Omega}$ be a local minimizer of f over $\boldsymbol{\Omega}$, and $\boldsymbol{d}$ a feasible direction at $\boldsymbol{x}^*$. ***If $\boldsymbol{d}^T \nabla f(\boldsymbol{x}^*) = 0$, then $\boldsymbol{H}(\boldsymbol{x}^*)$ is positive semidefinite****, where $\boldsymbol{H}(\boldsymbol{x}^*)$ is the Hessian of f at $\boldsymbol{x}^*$.*

Theorem 2 (**SONC, Interior Case**)

Let $f \in \mathcal{C}^2(\boldsymbol{\Omega})$ be a real-valued function defined on some set $\boldsymbol{\Omega} \subseteq \mathbb{R}^n$, whose boundary is denoted by $\boldsymbol{\partial\Omega}$. Let also $\boldsymbol{x}^* \in \boldsymbol{\Omega} \backslash \boldsymbol{\partial\Omega}$ be a local minimizer of f over $\boldsymbol{\Omega}$, then ***$\boldsymbol{H}(\boldsymbol{x}^*)$ is positive semidefinite****.*

Conditions for Local Minimizers

Remark 1 (Special Cases).

Boundary Case:

$$\text{If } \boldsymbol{d}^T \nabla f(\boldsymbol{x}^*) = 0, \text{ then } \boldsymbol{H}(\boldsymbol{x}^*) \geq 0 \xrightarrow{\text{in } \mathbb{R}} \text{If } f'(x^*) = 0, \text{ then } f''(x^*) \geq 0. \tag{1}$$

Interior Case:

$$\boldsymbol{H}(\boldsymbol{x}^*) \geq 0 \xrightarrow{\text{in } \mathbb{R}} f''(x^*) \geq 0. \tag{2}$$

Conditions for Local Minimizers

Example 1

Consider a function of one variable $f : \mathbb{R} \to \mathbb{R} : f(x) = x^3$. Show that the point $x = 0$ satisfies both the FONC and SONC.

Solution. Since $f'(0) = f''(0) = 0$, the point $x = 0$ satisfies both the FONC and SONC; though $x = 0$ is not a minimizer. Such a point is called **an inflection point**.

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Definition 1 (Inflection Point)

An inflection point of a single-variable function is **a stationary/critical point on the graph of the function at which the second derivative changes sign**.

- In particular, a point x^* is an inflection point for a single-variable function f if $f''(x^*) = 0$, and the lowest-order (above the second; e.g. third, fifth, etc.) non-zero derivative is odd. That is, $f'(x^*) = 0, f''(x^*) = 0, \ldots, f^{(k-1)}(x^*) = 0, f^{(k)}(x^*) \neq 0$ (k is an odd integer greater than 2).

- In the previous example, $f''(0) = 0$ and $f'''(0) = 6 \neq 0$; hence $x = 0$ is an inflection point.

Remark 2

The previous example shows that the necessary conditions, **the FONC and the SONC, are not sufficient**.

Conditions for Local Minimizers

Example 2

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(\boldsymbol{x}) = x_1^2 - x_2^2$. Show that $\boldsymbol{x} = [0, 0]^T$ is not a minimizer.

Solution. The FONC requires that $\nabla f(\boldsymbol{x}) = [2x_1, -2x_2]^T = \mathbf{0}$. Thus, $\boldsymbol{x} = [0, 0]^T$ satisfies the FONC. The Hessian matrix of f is

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Therefore, for any feasible direction $\boldsymbol{d} = [d_1, d_2]^T \in \mathbb{R}^2$, $\boldsymbol{d}^T \boldsymbol{H}(\boldsymbol{x})\boldsymbol{d} = 2\,d_1^2 - 2\,d_2^2$. Hence the Hessian matrix is **indefinite**, since $\boldsymbol{d}^T \boldsymbol{H}(\boldsymbol{x})\boldsymbol{d} \geq 0$ if $|d_1| \geq |d_2|$ (e.g., $\boldsymbol{d} = [1, 0]^T$), and $\boldsymbol{d}^T \boldsymbol{H}(\boldsymbol{x})\boldsymbol{d} < 0$ if $|d_1| < |d_2|$ (e.g., $\boldsymbol{d} = [0, 1]^T$). Thus, $\boldsymbol{x} = [0, 0]^T$ does not satisfy the SONC; hence it is not a minimizer. Such a point is called a **saddle point**.

Conditions for Local Minimizers

Definition 2 (Saddle Point)

A saddle point is **a point in the domain of a function that is a stationary point, but not a local extremum**.

- Mathematically, a stationary point $\boldsymbol{x}^*$ is a saddle point for a function $f(\boldsymbol{x})$ if **the Hessian $\mathrm{H}(\boldsymbol{x}^*)$ is indefinite**.
- The characteristic of a saddle point $\boldsymbol{x}^* \in \mathbb{R}^2$ is that it corresponds to a local minimum/maximum of $f(x_1, x_2)$ w.r.t. one variable, say, x_1 (the other variable being fixed at x_2^*) and a local maximum/minimum of $f(x_1, x_2)$ w.r.t. the second variable x_2 (the other variable being fixed at x_1^*).

Outline

Conditions for Local Minimizers

In this section we consider **the sufficient conditions** for the local minimum, local maximum, or inflection/saddle points of an **unconstrained multivariable function**.

Outline

Conditions for Local Minimizers

The following theorems provide the sufficient conditions that imply that $\boldsymbol{x}^*$ is a local minimizer, local maximizer, or an inflection/saddle point.

Theorem 3 (Second-Order Sufficient Condition (SOSC), Interior Case)

Let $f \in \mathcal{C}^2(\mathbb{R}^n), n \in \mathbb{Z}^+$, and $\boldsymbol{x}^* \in \mathbb{R}^n$. Suppose that,

1. $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$;
2. $\boldsymbol{H}(\boldsymbol{x}^*) > 0$ ($\boldsymbol{H}(\boldsymbol{x}^*)$ *is positive definite).*

Then, x^* is a strict local minimizer of f.

Theorem 4

Let $f \in \mathcal{C}^2(\mathbb{R}^n), n \in \mathbb{Z}^+$, and $\boldsymbol{x}^* \in \mathbb{R}^n$. Suppose that,

1. $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$;
2. $\boldsymbol{H}(\boldsymbol{x}^*) < 0$ ($\boldsymbol{H}(\boldsymbol{x}^*)$ *is negative definite).*

Then, x^* is a strict local maximizer of f.

Conditions for Local Minimizers

Theorem 5

Let $f \in \mathcal{C}^2(\mathbb{R}^n), n \in \mathbb{Z}^+$, *and* $\boldsymbol{x}^* \in \mathbb{R}^n$. *Suppose that,*

1. $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$;
2. $\boldsymbol{H}(\boldsymbol{x}^*)$ *is indefinite.*

Then, $\boldsymbol{x}^*$ *is a saddle point of* f.

Conditions for Local Minimizers

Example 3

Show that the point $\boldsymbol{x}^* = \mathbf{0}$ is a local minimum of the function $f(\boldsymbol{x}) = x_1^2 + x_2^2$.

Solution. We have $\nabla f(\boldsymbol{x}) = [2x_1, 2x_2]^T$; thus $\nabla f(\mathbf{0}) = \mathbf{0}$. For all $\boldsymbol{x} \in \mathbb{R}^2$, we have

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0.$$

The point $\boldsymbol{x} = \mathbf{0}$ satisfies the FONC, SONC, and SOSC, so it is a strict local minimizer. Actually, $\boldsymbol{x} = \mathbf{0}$ is a strict global minimizer of f.

Conditions for Local Minimizers

Example 4

Find the minimum point(s) of the function,

$$f(x) = \frac{1}{4}\,x^4 - 2\,x^3 + \frac{11}{2}\,x^2 - 6\,x + 1, \tag{3}$$

Solution. The FONC for a local minimum point is

$$f'(x^*) = {x^*}^3 - 6\,{x^*}^2 + 11\,x^* - 6 = (x^* - 1)\,(x^* - 2)\,(x^* - 3) = 0.$$

These equations are satisfied at the points $x_1^* = 1, x_2^* = 2$, and $x_3^* = 3$. To find the nature of these stationary points, we have to use the sufficiency condition:

$$f''(x^*) = 3\,{x^*}^2 - 12\,x^* + 11.$$

Conditions for Local Minimizers

The nature of the stationary points $x_i^*, i = 1, 2, 3$, is as given in the table below.

Point x_i^*	Value of $f''(x_i^*)$	Nature of x_i^*	Value of $f(x_i^*)$
1	2	Local minimum	-1.25
2	-1	Local maximum	-1
3	2	Local minimum	-1.25

Conditions for Local Minimizers

Example 5

Find the minimum point(s) of the function,

$$f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6.$$

Solution. The FONC for a local minimum point is

$$\nabla f(\boldsymbol{x}^*) = \left(3\, {x_1^*}^2 + 4\, x_1^*, 3\, {x_2^*}^2 + 8\, x_2^*\right)^T = (x_1^*\,(3\, x_1^* + 4), x_2^*(3\, x_2^* + 8))^T = \mathbf{0}.$$

These equations are satisfied at the points $\boldsymbol{x}_1^* = (0, 0)^T, \boldsymbol{x}_2^* = (0, -8/3)^T, \boldsymbol{x}_3^* = (-4/3, 0)^T$; $\boldsymbol{x}_4^* = (-4/3, -8/3)^T$. To find the nature of these stationary points, we have to use the sufficiency condition.

$$\boldsymbol{H}(\boldsymbol{x}^*) = \begin{pmatrix} 6\, x_1^* + 4 & 0 \\ 0 & 6\, x_2^* + 8 \end{pmatrix}.$$

Conditions for Local Minimizers

For any feasible direction $\boldsymbol{d} \in \mathbb{R}^2$,

$$\boldsymbol{d}^T \boldsymbol{H}(\boldsymbol{x}^*)\boldsymbol{d} = (6x_1^* + 4)d_1^2 + (6x_2^* + 8)d_2^2. \quad (4)$$

The nature of the stationary points $\boldsymbol{x}_i^*, i = 1, 2, 3, 4$, is as given in the table below.

Point $\boldsymbol{x}_i^*$	Value of $\boldsymbol{d}^T \boldsymbol{H}(\boldsymbol{x}_i^*)\boldsymbol{d}$	Nature of $\boldsymbol{H}$	Nature of $\boldsymbol{x}_i^*$	Value of $f(\boldsymbol{x}_i^*)$
$(0,0)^T$	$4d_1^2 + 8d_2^2$	Positive definite	Strict local minimum	6
$(0,-8/3)^T$	$4d_1^2 - 8d_2^2$	Indefinite	Saddle point	$418/27 \approx 15.48$
$(-4/3,0)^T$	$-4d_1^2 + 8d_2^2$	Indefinite	Saddle point	$194/27 \approx 7.19$
$(-4/3,-8/3)^T$	$-4d_1^2 - 8d_2^2$	Negative definite	Strict local maximum	$50/3 \approx 16.67$

Conditions for Local Minimizers

Example 6

The figure below shows two frictionless rigid bodies (carts) A and B connected by three linear elastic springs having spring constants k_1, k_2, and k_3. The springs are at their natural positions when the applied force P is zero. Find the displacements x_1 and x_2 under the force P by using the principle of minimum potential energy.

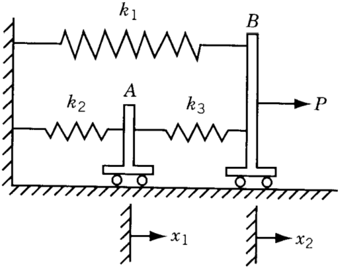

Figure 1: Spring-cart system.

Conditions for Local Minimizers

The principle of minimum total potential energy is a fundamental concept used in physics, chemistry, biology, and engineering, and it asserts that **a body shall displace to a position that minimizes the total potential energy**. Hence the spring-cart system will be in equilibrium under the load P if the total potential energy is a minimum.

The total potential energy of the system is given by,

Total potential energy $(U) =$ elastic potential energy of springs - work done by external forces acting on the structure

- The potential energy of a stretched spring is $\frac{1}{2}k\,x^2$, where k is the spring constant[1], and x is the amount of stretch (displacement or extension) of the spring from its rest position.
- The work done by the external force P is $P\,x_2$.

[1] A measure of the stiffness of a spring (large $k \to$ stiff spring, small $k \to$ soft spring).

Conditions for Local Minimizers

Hence,

$$U(x_1, x_2) = \left(\frac{1}{2}k_1 x_2^2 + \frac{1}{2}k_2 x_1^2 + \frac{1}{2}k_3 (x_2 - x_1)^2\right) - Px_2. \tag{5}$$

The necessary conditions for the minimum of U are

$$\left.\frac{\partial U}{\partial x_1}\right|_{\boldsymbol{x}=\boldsymbol{x}^*} = k_2 x_1^* - k_3(x_2^* - x_1^*) = 0,$$

$$\left.\frac{\partial U}{\partial x_2}\right|_{\boldsymbol{x}=\boldsymbol{x}^*} = k_3(x_2^* - x_1^*) + k_1 x_2^* - P = 0.$$

$$\Longrightarrow x_1^* = \frac{Pk_3}{k_1k_2 + k_1k_3 + k_2k_3}; \tag{6a}$$

$$x_2^* = \frac{P(k_2 + k_3)}{k_1k_2 + k_1k_3 + k_2k_3}. \tag{6b}$$

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The sufficiency conditions for the minimum at (x_1^*, x_2^*) can also be verified by testing the positive definiteness of the Hessian matrix of U. The Hessian matrix of U evaluated at (x_1^*, x_2^*) is

$$\boldsymbol{H}(\boldsymbol{x}^*) = \begin{bmatrix} \dfrac{\partial^2 U}{\partial x_1^2} & \dfrac{\partial^2 U}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 U}{\partial x_1 \partial x_2} & \dfrac{\partial^2 U}{\partial x_2^2} \end{bmatrix}_{(x_1^*, x_2^*)} = \begin{bmatrix} k_2 + k_3 & -k_3 \\ -k_3 & k_1 + k_3 \end{bmatrix}.$$

For any nonzero $\boldsymbol{d} = [d_1, d_2]^T \in \mathbb{R}^2$, we have

$$\boldsymbol{d}^T \mathbf{H}(\boldsymbol{x}^*)\boldsymbol{d} = d_1^2 k_2 + d_2^2 k_1 + (d_1 - d_2)^2 k_3 > 0,$$

since the spring constants are always positive. Thus, the matrix $\boldsymbol{H}(\boldsymbol{x}^*)$ is positive definite, and hence (x_1^*, x_2^*) corresponds to the minimum of potential energy.

Outline

2 Multivariable Unconstrained Optimization

- Second-Order Sufficient Condition (SOSC)
- Special Cases

Conditions for Local Minimizers

For some special functions, the Hessian vanishes at the stationary point.

- For such functions, second-order conditions are not sufficient to determine the nature of the point. Instead, **higher-order conditions need to be examined**. This is true for single as well as multivariable functions.

Conditions for Local Minimizers

- Consider a single-variable function $f(x)$ that has the following relations at the stationary point x^*:

$$\begin{aligned} f'(x^*) &= 0, \\ f''(x^*) &= 0, \\ &\vdots \\ f^{(k-1)}(x^*) &= 0, \\ f^{(k)}(x^*) &\neq 0, \quad k > 2. \end{aligned}$$

The following statements hold:

a) If k is odd, then x^* is an inflection point.

b) If k is even, then:

1. x^* is a point of local minimum if $f^{(k)}(x^*) > 0$.
2. x^* is a point of local maximum if $f^{(k)}(x^*) < 0$.