Operations Research
Chapter 4: Linear Programming

Dr. Kareem Elgindy

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Outline

Outline

Main Questions

In this chapter we shall try to answer the following FOUR main questions:

1. WHAT IS LINEAR PROGRAMMING (LP)?
2. WHAT IS THE GENERAL STANDARD MATHEMATICAL FORMULATION OF LP PROBLEMS?
3. HOW TO TRANSFORM A LP PROBLEM INTO THE GENERAL STANDARD MATHEMATICAL FORMULATION OF LP PROBLEMS?
4. HOW TO SOLVE TWO-DIMENSIONAL LP PROBLEMS GRAPHICALLY?

Outline

Introduction

- The oil industry long has used LP in **refinery planning**, as it determines how much of its raw product should become different grades of gasoline and how much should be used for petroleum-based byproducts.

- LP is used to solve the **diet problem**: Find the cheapest combination of foods that will satisfy all your nutritional requirements.

- LP is used also in **scheduling**, **transportation**, **manufacturing**, **revenue management**, **telecommunications**, **advertising**, **architecture**, **circuit design**, and countless other areas.

- LP is considered a revolutionary development that permits us to make optimal decisions in complex situations. At least **four Nobel Prizes** were awarded for contributions related to LP.

- Some of the major contributions in this subject are credited to Leonid Kantorovich (19 January 1912 – 7 April 1986), Tjalling Koopmans (August 28, 1910 – February 26, 1985), George Dantzig (November 8, 1914 – May 13, 2005), and John von Neumann (December 28, 1903 – February 8, 1957).

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WHAT IS LP? WHAT IS AN LP PROBLEM?

Definition (LP)

LP is the process of solving a linear optimization problem, where **all of the constraints and the objective function are linear**. Linear indicates that no variables are raised to higher powers, such as squares, and there is no multiplication of variables.

Definition (An LP Problem)

An LP problem is **an optimization problem in which all of the constraints and the objective function are linear**.

Outline

GENERAL STANDARD MATHEMATICAL FORMULATION OF LP PROBLEMS

The general standard mathematical LP problem can be stated as,

$$\begin{aligned}
\text{minimize } & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n && \text{(1a)}\\
\text{subject to } & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1, && \text{(1b)}\\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2, && \text{(1c)}\\
& \vdots && \text{(1d)}\\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m, && \text{(1e)}\\
& x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0. && \text{(1f)}
\end{aligned}$$

where $m < n$, and $b_i \geq 0 \,\forall i$.

GENERAL STANDARD MATHEMATICAL FORMULATION OF LP PROBLEMS

Let $\boldsymbol{c} = [c_1, c_2, \ldots, c_n]^T \in \mathbb{R}^n, \mathbf{A} = (a_{i,j}) \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} = [b_1, b_2, \ldots, b_m]^T \in \mathbb{R}^m, m < n$, rank $\mathbf{A} = m$, and $\boldsymbol{b} \geq \mathbf{0}$. Then the general standard mathematical LP problem can also be written in matrix notation as,

$$\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \tag{2a}$$

$$\text{subject to } \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \tag{2b}$$

$$\boldsymbol{x} \geq \mathbf{0}. \tag{2c}$$

Any solution that satisfies the constraints (2b) and (2c) is called a feasible solution.

Outline

IMPORTANT OBSERVATIONS ABOUT STANDARD LP MODELS

- It can be seen that there are m equations in n decision variables in a LP problem. We can assume that $m < n$; for:
  - if $m > n$, there would be $m - n$ redundant equations that could be eliminated, or the equations are inconsistent.
  - The case $n = m$ is of no interest, for then there is either a unique solution $\boldsymbol{x}$ that satisfies Eqs. (2b) and (2c) (in which case there can be no optimization), or no solution, in which case the constraints are inconsistent.
  - The case $m < n$ corresponds to an underdetermined set of linear equations, which, if they have one solution, have an infinite number of solutions.

The problem of LP is to find one of these solutions that satisfies Eqs. (2b) and (2c) and yields the minimum of $\boldsymbol{c}^T\boldsymbol{x}$.

IMPORTANT OBSERVATIONS ABOUT STANDARD LP MODELS

- Several variations of the standard LP problem are possible; for example, **instead of minimizing, we can maximize**.

- The equality constraints may be in the form of inequalities such as "$\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$," or "$\mathbf{A}\boldsymbol{x} \geq \boldsymbol{b}$," or one or more of the unknown variables is not required to be nonnegative (free). In fact, as we shall see next, these variations can all be rewritten into the standard form shown before.

LP PROBLEMS WITH INEQUALITY CONSTRAINTS

A LP problem with inequality constraints, say $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$, can be converted into the standard form by using the **slack variables** (or more loosely, **slacks**) $y_i, i = 1, \ldots, m$. In particular, we can rewrite $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$ as,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + y_1 = b_1, \tag{3a}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + \quad + y_2 = b_2, \tag{3b}$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + \quad + y_m = b_m, \tag{3c}$$

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0. \tag{3d}$$

LP PROBLEMS WITH INEQUALITY CONSTRAINTS

By considering the problem as one having $n + m$ unknowns $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$, the $m \times (n + m)$ matrix that now describes the linear equality constraints is of the special form $[\mathbf{A}, \mathbf{I}_m]$ (that is, its columns can be partitioned into two sets; the first n columns make up the original $\mathbf{A}$ matrix and the last m columns make up an $m \times m$ identity matrix).

LP PROBLEMS WITH INEQUALITY CONSTRAINTS

Hence, in more compact notation, the LP problem formulation can be represented as:

$$\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \tag{4a}$$

$$\text{subject to } \mathbf{A}\boldsymbol{x} + \mathbf{I}_m \boldsymbol{y} = [\mathbf{A}, \mathbf{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \tag{4b}$$

$$\boldsymbol{x} \geq \mathbf{0}, \boldsymbol{y} \geq \mathbf{0}, \tag{4c}$$

where $\boldsymbol{y}$ is the vector of slack variables.

LP PROBLEMS WITH INEQUALITY CONSTRAINTS

If the inequality sign is reversed so that a typical inequality is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i, \tag{5}$$

then it is clear that this is equivalent to,

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - y_i = b_i; \tag{6a}$$

$$y_i \geq 0. \tag{6b}$$

Variables, such as y_i, adjoined in this fashion to convert a "greater than or equal to" inequality to equality are called **surplus variables**[1].

[1] Called a "surplus" variable, because it is the amount (surplus) by which the left side of the inequality exceeds the right side.

LP PROBLEMS WITH INEQUALITY CONSTRAINTS

The LP problem formulation can now be represented as:

$$\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \tag{7a}$$

$$\text{subject to } \mathbf{A}\boldsymbol{x} - \mathbf{I}_m \boldsymbol{y} = [\mathbf{A}, -\mathbf{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \tag{7b}$$

$$\boldsymbol{x} \geq \mathbf{0}; \boldsymbol{y} \geq \mathbf{0}. \tag{7c}$$

Remark 1.

Note that neither slack nor surplus variables contribute to the objective function $c^T x$.

LP PROBLEMS WITH FREE VARIABLES

If a LP problem is given in standard form except that **one or more of the unknown variables is not required to be nonnegative**, the problem can be transformed to standard form by the following **simple technique**: Suppose that the restriction $x_1 \geq 0$ is not present and hence x_1 is free to take on either positive or negative values. Take any one of the m equations in (2b) which has a nonzero coefficient for x_1. Say, for example,

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i, \tag{8}$$

where $a_{i1} \neq 0$. Then x_1 can be expressed as a linear combination of the other variables plus a constant; i.e.,

$$x_1 = \left(b_i - \sum_{j=2}^{n} a_{ij}x_j\right) / a_{i1}. \tag{9}$$

LP PROBLEMS WITH FREE VARIABLES

If this expression is substituted for x_1 everywhere in (2a) and (2b), we are led to a new problem of exactly the same form but expressed in terms of the variables $x_2, x_3, \ldots, x_n$ only.

LP PROBLEMS WITH FREE VARIABLES

As a result of this simplification, we obtain a standard LP problem having $n-1$ variables and $m-1$ constraint equations. The value of the optimal variable x_1^* can be determined after solution through (9).

LP PROBLEMS WITH FREE VARIABLES

Example

As a specific instance of the above technique consider the problem:

$$\text{minimize } x_1 + 3x_2 + 4x_3 \tag{10a}$$

$$\text{subject to } x_1 + 2x_2 + x_3 = 5, \tag{10b}$$

$$2x_1 + 3x_2 + x_3 = 6, \tag{10c}$$

$$x_2 \geq 0, \quad x_3 \geq 0. \tag{10d}$$

Since x_1 is free, we solve for it from the first constraint, obtaining

$$x_1 = 5 - 2x_2 - x_3. \tag{11}$$

LP PROBLEMS WITH FREE VARIABLES

Substituting this into the objective and the second constraint, we obtain the equivalent problem (subtracting five from the objective):

$$\text{minimize } x_2 + 3x_3 \tag{12a}$$

$$\text{subject to } x_2 + x_3 = 4, \tag{12b}$$

$$x_2 \geq 0, \quad x_3 \geq 0, \tag{12c}$$

which is a problem in standard form. After the smaller problem is solved (the answer is $x_2^* = 4, x_3^* = 0$) the value for $x_1^* (x_1^* = -3)$ can be found from (11).

Outline

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

A LP problem with **only two variables** presents a simple case for which the solution can be obtained by using a rather elementary **graphical method**. Apart from the solution, many fundamental concepts of LP are easily illustrated in two-dimensional space. Therefore, we consider linear problems in $\mathbb{R}^2$ before discussing general LP problems.

The graphical procedure includes two steps:

1. **Determination of the feasible solution space.**
2. **Determination of the optimum solution from among all the feasible points in the solution space.**

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

The graphical method works for any general two-dimensional LP problem, **not necessarily in the standard mathematical formulation of LP problems**.

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Example

Consider the following LP problem,

$$\text{maximize } \boldsymbol{c}^T \boldsymbol{x} \tag{13a}$$

$$\text{subject to } \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \tag{13b}$$

$$\boldsymbol{x} \geq \mathbf{0}, \tag{13c}$$

where $\boldsymbol{x} = (x_1, x_2)^T, \boldsymbol{c}^T = (1, 5), \boldsymbol{b}^T = (30, 12)$;

$$\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}.$$

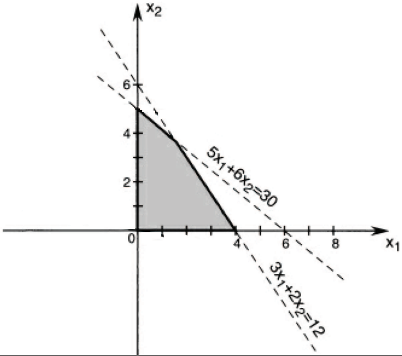
TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

- First, we account for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. The nonnegativity of the variables restricts the solution-space area to the **first quadrant** that lies above the x_1-axis and to the right of the x_2-axis.

- To account for the remaining two constraints, first replace each inequality with an equation and then graph the resulting straight line.
  - Next, consider the effect of the inequality. To determine the correct side, choose $(0, 0)$ as a reference point. Because 0 is less than 30 and 12, the half-space representing the inequality includes the origin.

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Now let us sketch the constraints in $\mathbb{R}^2$. The region of feasible points is depicted by the shaded region in the figure below.



TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Afterward, notice that the set of points,

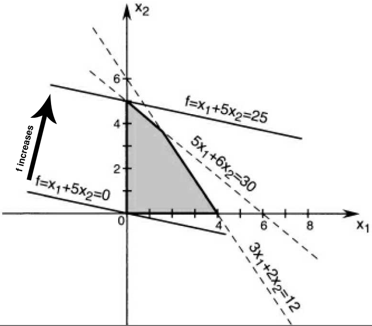
$$\{(x_1, x_2) : \boldsymbol{c}^T \boldsymbol{x} = x_1 + 5x_2 = f, f \in \mathbb{R}\},$$

specifies a family of parallel straight lines in $\mathbb{R}^2$. Each member of this family can be obtained by setting f equal to some real number.

The question is then how big f can become so that the straight line $x_1 + 5x_2 = f$ meets the feasible region somewhere.

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Graphically, it is not hard to realize that the optimal point corresponds to the vertex $(x_1^*, x_2^*) = (0, 5)$, and the optimal objective value is $f^* = 50$.



TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Note that regardless of what the objective is, as long as it is linear, **the optimum value generally occurs at an extreme point or vertex of the feasible region**. The vertices of the feasible region play a crucial role in the understanding of LP problems, as we shall see.

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Remark 2.

In some cases, when using the graphical method, the optimal straight line $f^* = \boldsymbol{c}^T\boldsymbol{x}^*$ may be **parallel to one boundary/side of the polygon boundary of the feasible region**. In this case **all of the intersection points** will yield the same value for the objective function $\boldsymbol{c}^T\boldsymbol{x}^*$, and therefore any one of them is a solution. That is, no unique solution exists; instead, there is **an infinite number of optimal solutions**.

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Example

Suppose that you are given two different types of concrete. The first type contains 30% cement, 40% gravel, and 30% sand (all percentages of weight). The second type contains 10% cement, 20% gravel, and 70% sand. The first type of concrete costs $5 per pound and the second type costs $1 per pound. How many pounds of each type of concrete should you buy and mix together so that your cost is minimized but you get a concrete mixture that has at least a total of 5 pounds of cement, 3 pounds of gravel, and 4 pounds of sand?

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

DEFINITION OF DESIGN VARIABLES: The decision problem is to determine the mass (in pounds) of each type of concrete that will minimize the total cost. Therefore, the decision variables are identified and defined as follows:

x is the mass of the first type concrete.
y is the mass of the second type concrete.

OPTIMIZATION CRITERION: The decision objective is to minimize the cost, which depends on the concrete mixture mass. Hence, the optimization criterion is the total cost defined by:

$$\text{Cost function} = f = 5\,x + y.$$

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

FORMULATION OF CONSTRAINTS: The constraint for the problem is based on the structure of the concrete mixture:

$$0.3\,x + 0.1\,y \geq 5, \tag{14a}$$

$$0.4\,x + 0.2\,y \geq 3, \tag{14b}$$

$$0.3\,x + 0.7\,y \geq 4. \tag{14c}$$

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

The problem can therefore be represented as,

$$\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \tag{15a}$$

$$\text{subject to } \mathbf{A}\boldsymbol{x} \geq \boldsymbol{b}, \tag{15b}$$

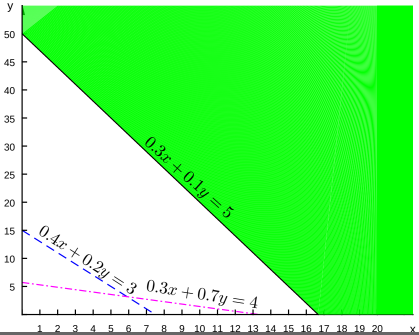
$$\boldsymbol{x} \geq \mathbf{0}, \tag{15c}$$

where $\boldsymbol{c}^T = (5, 1), \boldsymbol{b} = (5, 3, 4)^T$;

$$\mathbf{A} = \begin{pmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}.$$

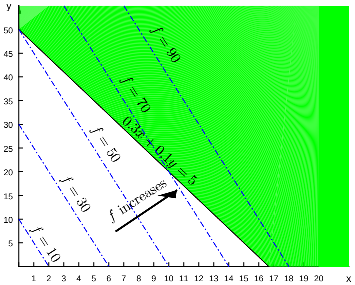
TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Now let us sketch the constraints in $\mathbb{R}^2$. The region of feasible points is depicted by the shaded region in the figure below.



TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Using the graphical method, we get a solution of $(x^*, y^*) = (0, 50)$ with the optimal objective value $f^* = 25$. This means that we should purchase 50 pounds of the second type of concrete.



TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Remark 3.

The graphical method of solution may be extended to a case in which there are three variables. In this case, **each constraint is represented by a plane** in three dimensions, and the feasible region bounded by these planes is a **polyhedron**.

TWO-DIMENSIONAL LP Problems– GRAPHICAL LP SOLUTION

Remark 4.

The graphical method shown before can be used for two/three-dimensional problems; however, **real-life LP problems consist of many variables**, and to solve these LP problems, one has to resort to a **numerical optimization method** such as the simplex method.