Operations Research
Chapter 4: Linear Programming

Dr. Kareem Elgindy

28/04/2016

Outline

1 Basic Solutions

Outline

1 Basic Solutions

Preliminary: Linear Dependence

Definition (LINEAR DEPENDENCE)

A set of vectors is said to be linearly dependent **if one of the vectors in the set can be defined as a linear combination of the other vectors**. If no vector in the set can be written in this way, then the vectors are said to be linearly independent. Mathematically speaking, a distinct number of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, in a vector space $\mathbb{V}$ are said to be linearly dependent, if there exists a finite number of scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 + \cdots + a_n \boldsymbol{v}_n = \mathbf{0}. \tag{1}$$

Preliminary: Linear Dependence

Definition (LINEAR DEPENDENCE)

The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, are said to be linearly independent if Eq. (1) implies that $a_i = 0, i = 1, \ldots, n$. This implies that no vector in the set can be represented as a linear combination of the remaining vectors in the set.

Preliminary: Linear Dependence

For a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, the following statements are equivalent:

1. The rows of $\mathbf{A}$ are linearly independent.
2. The columns of $\mathbf{A}$ are linearly independent.
3. $\mathbf{A}$ is non-singular.
4. $\det(\mathbf{A}) \neq 0$.
5. $\operatorname{rank}(\mathbf{A}) = n$.

Preliminary: Linear Dependence

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 7 & 6 \\ 2 & 6 & 4 \\ 4 & 3 & -1 \end{pmatrix}$. Determine whether the columns of $\mathbf{A}$ are linearly dependent or independent.

Solution

Since $det(\mathbf{A}) = 0$, *the columns of* $\mathbf{A}$ *are linearly dependent.*

Preliminary: Linear Dependence

Another Solution: Let $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$ denote the columns of $\mathbf{A}$, respectively. These vectors are **linearly dependent** if and only if there is a nonzero vector $\boldsymbol{x} = [x_1, x_2, x_3]^T$ such that:

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 = \mathbf{0}. \tag{2}$$

The column vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$ are **linearly independent** if and only if the condition (2) implies $x_1 = x_2 = x_3 = 0$. Now, Eq. (2) is equivalent to the homogeneous system,

$$\mathbf{A}\boldsymbol{x} = \mathbf{0}. \tag{3}$$

The next step is to perform elementary row operations on the coefficient matrix $\mathbf{A}$ to put it in its reduced row echelon form.

Preliminary: Linear Dependence

In particular, since

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 6 \\ 2 & 6 & 4 \\ 4 & 3 & -1 \end{pmatrix} \xrightarrow[(R_3)\leftarrow(R_3-4\,R_1)]{(R_2)\leftarrow(R_2-2\,R_1)} \begin{pmatrix} 1 & 7 & 6 \\ 0 & -8 & -8 \\ 0 & -25 & -25 \end{pmatrix}$$

$$\xrightarrow[(R_3)\leftarrow(R_3+25\,R_2)]{(R_2)\leftarrow-\frac{1}{8}(R_2),\ (R_1)\leftarrow(R_1-7\,R_2)} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Preliminary: Linear Dependence

It follows that the system $\mathbf{A}x = \mathbf{0}$, is equivalent to the following system:

$$x_1 - x_3 = 0, \tag{4a}$$
$$x_2 + x_3 = 0, \tag{4b}$$
$$0 = 0. \tag{4c}$$

In this system, the variable x_3 is free, so a nontrivial solution of the system can be obtained by substituting any nonzero number for x_3, and backsolving then for x_2 and x_1. For example, setting $x_3 = k$, for some parameter $k \in \mathbb{R}$, the family of solutions of the system is given by $\boldsymbol{x} = [k, -k, k]^T$.

Preliminary: Linear Dependence

- If we use this general solution of the system, we can express $\boldsymbol{a}_3$ as a linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, as follows:

$$\boldsymbol{a}_3 = -\boldsymbol{a}_1 + \boldsymbol{a}_2. \tag{5}$$

Hence the columns of $\mathbf{A}$ are **linearly dependent**.

Basic Solutions

In the following discussion **we only consider LP problems in standard form**.

- Consider the system of equalities $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m, m < n$, rank $(\mathbf{A}) = m$, and $\mathbf{0} \leq \boldsymbol{x} \in \mathbb{R}^n$. Let $\mathbf{B}$ be a square matrix whose columns are m linearly independent columns of $\mathbf{A}$. If necessary, we reorder the columns of $\mathbf{A}$ so that the columns in $\mathbf{B}$ appear first: $\mathbf{A}$ has the form $\mathbf{A} = [\mathbf{B}, \mathbf{D}]$, where $\mathbf{D}$ is an $m \times (n - m)$ matrix whose columns are the remaining columns of $\mathbf{A}$.

- The matrix $\mathbf{B}$ is **nonsingular**, and thus we can solve the equation $\mathbf{B}\boldsymbol{x}_{\mathbf{B}} = \boldsymbol{b}$ for the m-vector $\boldsymbol{x}_{\mathbf{B}}$. The solution is $\boldsymbol{x}_{\mathbf{B}} = \mathbf{B}^{-1}\boldsymbol{b}$.

Basic Solutions

- Let $\boldsymbol{x}$ be the n-vector whose first m components are equal to $\boldsymbol{x}_{\mathbf{B}}$ and the remaining components are equal to zero; that is, $\boldsymbol{x} = [\boldsymbol{x}_{\mathbf{B}}^T, \mathbf{0}^T]^T$. Then, $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

Basic Solutions

Definition (BASIC SOLUTION)

We call $[x_{\mathbf{B}}^T, \mathbf{0}^T]^T$ a basic solution to $\mathbf{A}x = b$ with respect to the **basis B**.

- Notice that a basic solution is one in which $n - m$ variables are set equal to zero.
- We refer to the components of the vector $x_{\mathbf{B}}$ as **basic variables** and the columns of **B** as **basic columns**.
- If some of the basic variables of a basic solution are zeros, then the basic solution is said to be a **degenerate basic solution**.
- The total number of possible basic solutions (the maximum number of corner points) is at most $\binom{n}{m}$.

Basic Solutions

Remark 1.

Assuming that:

1. the $m \times n$ matrix $\mathbf{A}$ has $m < n$, and
2. $rank(\mathbf{A}) = m$,

the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ will always have a solution and, in fact, it **will always have at least one basic solution**.

Basic Solutions

Remark 2.

- A vector $\boldsymbol{x}$ satisfying $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}$, is said to be a feasible solution.
- A feasible solution that is also basic is called **a basic feasible solution**.
- An optimal feasible solution that is basic is said to be **an optimal basic feasible solution**.
- If the basic feasible solution is a degenerate basic solution, then it is called **a degenerate basic feasible solution**.

Basic Solutions

Example

Consider the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbf{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 8 \\ 2 \end{pmatrix},$$

where $\boldsymbol{a}_i$ denotes the ith column of the matrix $\mathbf{A}$. Then, $\boldsymbol{x} = (6, 2, 0, 0)^T$ is a basic feasible solution with respect to the basis $\mathrm{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2]$, $\boldsymbol{x} = (0, 0, 0, 2)^T$ is a degenerate basic feasible solution with respect to the basis $\mathrm{B} = [\boldsymbol{a}_3, \boldsymbol{a}_4]$ (as well as $[\boldsymbol{a}_1, \boldsymbol{a}_4]$ and $[\boldsymbol{a}_2, \boldsymbol{a}_4]$), $\boldsymbol{x} = (3, 1, 0, 1)^T$ is a feasible solution that is not basic, and $\boldsymbol{x} = (0, 2, -6, 0)^T$ is a basic solution with respect to the basis $\mathrm{B} = [\boldsymbol{a}_2, \boldsymbol{a}_3]$, but is not feasible.

Basic Solutions

Example

Find all basic solutions of the system of linear equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where,

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 & -1 \\ 4 & 1 & 1 & -2 \end{pmatrix}; \boldsymbol{b} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}.$$

Solution

We now find all solutions of this system. We form the augmented matrix $[\mathbf{A}, \boldsymbol{b}]$ *of the system:*

$$[\mathbf{A}, \boldsymbol{b}] = \begin{pmatrix} 2 & 3 & -1 & -1 & -1 \\ 4 & 1 & 1 & -2 & 9 \end{pmatrix}.$$

Basic Solutions

Using elementary row operations, we transform this matrix into the reduced row echelon form given by,

$$[A, b] \xrightarrow{(R_1) \leftarrow \frac{1}{2}(R_1)} \begin{pmatrix} 1 & 3/2 & -1/2 & -1/2 & -1/2 \\ 4 & 1 & 1 & -2 & 9 \end{pmatrix}$$
$$\xrightarrow{(R_2) \leftarrow -4(R_1)+(R_2)} \begin{pmatrix} 1 & 3/2 & -1/2 & -1/2 & -1/2 \\ 0 & -5 & 3 & 0 & 11 \end{pmatrix}$$
$$\xrightarrow{(R_2) \leftarrow -\frac{1}{5}(R_2)} \begin{pmatrix} 1 & 3/2 & -1/2 & -1/2 & -1/2 \\ 0 & 1 & -3/5 & 0 & -11/5 \end{pmatrix}$$
$$\xrightarrow{(R_1) \leftarrow -\frac{3}{2}(R_2)+(R_1)} \begin{pmatrix} 1 & 0 & 2/5 & -1/2 & 14/5 \\ 0 & 1 & -3/5 & 0 & -11/5 \end{pmatrix}.$$

Basic Solutions

The corresponding system of equations is given by

$$x_1 + \frac{2}{5}x_3 - \frac{1}{2}x_4 = \frac{14}{5} \qquad (6a)$$

$$x_2 - \frac{3}{5}x_3 = -\frac{11}{5}. \qquad (6b)$$

Hence,

$$x_1 = \frac{14}{5} - \frac{2}{5}s + \frac{1}{2}t, \qquad (7a)$$

$$x_2 = -\frac{11}{5} + \frac{3}{5}s, \qquad (7b)$$

$$x_3 = s, \qquad (7c)$$

$$x_4 = t, \qquad (7d)$$

where s and t are arbitrary real numbers.

Basic Solutions

Using vector notation, we may write the system of equations above as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 14/5 \\ -11/5 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2/5 \\ 3/5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that we have **infinitely many solutions** parameterized by $s, t \in \mathbb{R}$. The total number of possible basic solutions is at most,

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!} = \frac{4!}{2!(4-2)!} = 6.$$

Basic Solutions

To find basic solutions that are feasible, we check each of the basic solutions for feasibility.

1. Our first candidate for a basic feasible solution is obtained by setting $x_3 = x_4 = 0$, which corresponds to the basis $\mathbf{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2]$. Solving $\mathbf{B}\boldsymbol{x}_{\mathbf{B}} = \boldsymbol{b}$, we obtain $\boldsymbol{x}_{\mathbf{B}} = (14/5, -11/5)^T$, and hence $\boldsymbol{x} = (14/5, -11/5, 0, 0)^T$ is **a basic solution that is not feasible**.

2. For our second candidate basic feasible solution, we set $\boldsymbol{x}_2 = \boldsymbol{x}_4 = 0$. We have the basis $\mathbf{B} = [\boldsymbol{a}_1, \boldsymbol{a}_3]$. Solving $\mathbf{B}\boldsymbol{x}_{\mathbf{B}} = \boldsymbol{b}$ yields $\boldsymbol{x}_{\mathbf{B}} = (4/3, 11/3)^T$. Hence, $\boldsymbol{x} = (4/3, 0, 11/3, 0)^T$ is **a basic feasible solution**.

Basic Solutions

3. A third candidate basic feasible solution is obtained by setting $x_1 = x_4 = 0$. We have a basis $\mathbf{B} = [\boldsymbol{a}_2, \boldsymbol{a}_3]$, resulting in $\boldsymbol{x} = (0, 2, 7, 0)^T$, which is **a basic feasible solution**.

4. We get our fourth candidate for a basic feasible solution by setting $\boldsymbol{x}_2 = \boldsymbol{x}_3 = 0$. However, the matrix,

$$\mathbf{B} = [\boldsymbol{a}_1, \boldsymbol{a}_4] = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$$

is **singular**. Therefore, $\mathbf{B}$ **cannot be a basis**, and we do not have a basic solution corresponding to $\mathbf{B} = [\boldsymbol{a}_1, \boldsymbol{a}_4]$.

Basic Solutions

5 Our fifth candidate for a basic feasible solution corresponds to setting $x_1 = x_3 = 0$, with the basis $\mathbf{B} = [\boldsymbol{a}_2, \boldsymbol{a}_4]$. This results in $\boldsymbol{x} = (0, -11/5, 0, -28/5)^T$, which is **a basic solution that is not feasible**.

6 Finally, the sixth candidate for a basic feasible solution is obtained by setting $x_1 = x_2 = 0$. This results in the basis $\mathbf{B} = [\boldsymbol{a}_3, \boldsymbol{a}_4]$, and $\boldsymbol{x} = (0, 0, 11/3, -8/3)^T$, which is **a basic solution that is not feasible**.

Basic Solutions

Hence, **we have 5 basic solutions, but only 2 of them are feasible**.

Theorem (THE FUNDAMENTAL THEOREM OF LP)

Consider a linear program in standard form. Then:

1. ***If there exists a feasible solution, then there exists a basic feasible solution.***
2. ***If there exists an optimal feasible solution, then there exists an optimal basic feasible solution.***

Basic Solutions

Example

Solve the following LP problem:

$$\text{Minimize } [2, -3, 9, 1]\boldsymbol{x} \tag{8a}$$

$$s.t. \begin{pmatrix} 2 & 3 & -1 & -1 \\ 4 & 1 & 1 & -2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}, \tag{8b}$$

$$\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T \geq 0. \tag{8c}$$

Basic Solutions

Solution

In the previous example, we found that there are only 2 basic feasible solutions, namely, $\mathbf{x}_1 = (4/3, 0, 11/3, 0)^T$*, and* $\mathbf{x}_2 = (0, 2, 7, 0)^T$*. Since,* $f(\mathbf{x}_1) = 107/3 \approx 35.7 < 57 = f(\mathbf{x}_2)$*, then* $\mathbf{x}^* = \mathbf{x}_1 = (4/3, 0, 11/3, 0)^T$ *is the optimal solution.*