Chapter 1
Error Analysis

Dr. Kareem Elgindy

Lecturer
Mathematics Department, Faculty of Science
Assiut University

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Outline

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Chapter Questions

In this chapter we shall try to answer the following main THREE questions:

1. **WHAT ARE THE TYPES OF ERRORS IN SCIENTIFIC COMPUTATIONS?**
2. **HOW CAN WE MEASURE THEM?**
3. **HOW CAN WE USE THE ANALYTIC FORM OF THE ERROR TO PREDICT AN ERROR UPPER BOUND?**

Outline

Introduction

- There is a fundamental concern in numerical analysis with the **error**, its
  1. **type**,
  2. **quantity (size)**;
  3. **analytic form**.

  In fact, for any particular problem, it is not enough simply to develop a method which we hope will give a reasonable approximation to the solution; we need to know:
  1. that it is **indeed giving us such an approximation**;
  2. **how good that approximation is**.

  Generally, a numerical analyst would want to understand the nature of the error in the computed solution.
  - Moreover, understanding the form of the error may allow one to **estimate it**, **minimize it**, or **estimate its upper bound**.

Introduction

Definition 1 (Error Analysis)

The study of the **kind**, **quantity**, and **analytic form** of the error that occurs, particularly in the field of numerical analysis, is called **error analysis**.

- The most fundamental feature of scientific computing is the **inevitable presence of error**, and the result of any scientific computation will be only **approximate**, in general.
  - For instance, *all measurements of physical quantities are subject to **errors (uncertainties)***. Moreover, *errors are encountered when attempting to **represent a number using a finite string of digits***.
- It is good, of course, to make the error as small as possible, but it is always there, and our general quest is to **ensure that the resulting error be tolerably small**.

Introduction

- When a quantity is measured, the outcome depends on the **measuring system**, **the measurement procedure**, **the skill of the operator**, **the environment**, and other effects. In order to draw valid conclusions, **the error must be indicated**.
- Eventually, the result of any scientific computation (such as a physical measurement) has two essential components:
  1. **A numerical value** (in a specified system of units) giving the best estimate possible of the quantity measured.
  2. The **degree of error** associated with this estimated value.
- There are many sources of approximation or inexactness in scientific computations. Some of these occur even **before computation begins** while others occur **during computation**. The accuracy of the final results of a computation may reflect a combination of any or all of these approximations.

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Types of Errors

- The majority of interacting systems in the real world are far too complicated to model in their entirety, i.e. **formulating the mathematical model is usually not an easy task and it may be more difficult than solving the model itself**.
  - Hence the first level of compromise is to identify the most important parts of the system. These will be included in the model, the rest will be excluded to facilitate the mathematical modeling.

Definition 2 (**Mathematical Model Error** (Before Computation Begins))

The error arising because of inaccuracy in the mathematical description of a real process is called the ***mathematical model error***.

Types of Errors

- Data errors limit the accuracy of any subsequent calculations involved in the numerical method applied for the solution of the model. In fact, ***the accuracy of the solutions obtained by the numerical method does not exceed the accuracy of the data itself***.

Definition 3 (**Input-Data Error** (Before Computation Begins))

The error arising from inaccuracy in specifying the initial data is called the ***input-data error*** (or simply ***data error***).

Types of Errors

The error made by truncating an infinite sum and approximating it by a finite sum is called **the truncation error**.

- In other words, the truncation error results from ignoring all but a finite number of terms of an infinite series.

Definition 4 (**The Truncation Error** (During Computation))

In general, we define ***the truncation error*** as **the difference between the true result (for the actual input) and the result produced by a given algorithm using exact arithmetic**.

Types of Errors

A real number may have an infinite number of digits in its decimal representation, and it does not end with an infinitely repeating pattern of digits. In practice, such a number must be represented using a finite string of digits. The error produced from such an operation is known as the **round-off error**.

- In certain types of computation, round-off error can be magnified as any initial errors are carried through one or more intermediate steps.

Definition 5 (**Round-Off Error** (During Computation))

- **The difference between the exact (correct) value of a number and its approximate representation in a certain number system using a finite number of digits** is called the ***round-off error***,

Outline

Size of Errors

Definition 6 (The Approximation Error)

The difference (discrepancy) between the exact mathematical value (true value) and its estimate, or approximation, is called **the approximation error**.

- In other words, if p is the exact value of a certain quantity, and p^* is its approximate value, then the difference $(p - p^*)$ is said to be the approximation error.

The true error is commonly expressed as an absolute value, i.e. $|p - p^*|$, and referred to as the **absolute error**. If we denote the absolute error in the approximation of p by E_a, then we can write

$$E_a = |p - p^*| . \tag{1}$$

Size of Errors

Clearly, an error of two centimeters is much more significant if we are measuring a rivet than a bridge.

- Hence, a shortcoming of the definition of absolute error is that **it takes no account of the order of magnitude of the value under examination**.

One way to account for the magnitudes of the quantities being evaluated is to **normalize** the error to the true value, as in

$$\frac{|p - p^*|}{|p|},$$

provided that $p \neq 0$. This error is referred to as the **relative error**, and is denoted by E_r, i.e.

$$\boxed{E_r = \frac{E_a}{|p|} = \frac{|p - p^*|}{|p|}.} \quad (2)$$

Size of Errors

Absolute Error Vs. Relative Error

1. The absolute error E_a is simply **the difference between the true value and the approximate value**, whereas the relative error E_r **expresses the error as a percentage of the true value**.
2. The the relative error E_r **takes account of the order of magnitude of the value** under examination, but the absolute error E_a **does not**.

Size of Errors

Example 1

Determine the absolute errors, relative errors, and the quality of approximations when approximating p by p^* in the following cases:

(a) $p = 1.1$ and $p^* = 1.11$.

(b) $p = 10,000$ and $p^* = 9999$.

(c) $p = 0.000012$ and $p^* = 0.000009$.

Solution 1

(a) $E_a = |p - p^*| = 0.01;\ E_r = \frac{|p-p^*|}{|p|} = 0.00\overline{909} = 0.9\overline{09}\%$, *(**good approximation**)*
where the bar over 09 *in* $0.00\overline{909}$ *is used to indicate that these digits repeat indefinitely.*

(b) $E_a = 1;\ E_r = 0.0001 = 0.01\%$ *(**good approximation**).*

(c) $E_a = 0.000003;\ E_r = 0.25 = 25\%$ *(**bad approximation**).*

Size of Errors

Important Observation

For some purposes, the absolute error E_a may be what we want; but in many cases, **the relative error E_r gives a better measure for the error in the approximation**.

- Generally, E_a **may be misleading as $|p|$ moves away from 1**, whereas, the relative error, E_r, is a better indicator for the accuracy of the approximation.

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Truncation Error

Consider the exponential function e^x, which may be expressed as the sum of the infinite series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots . \tag{3}$$

Stopping the calculation after any finite value of n will give an approximation to the value of e^x that will be in error (truncation error), but this error can be made as small as desired by making n large enough.

- One of the very basic examples of expressing the truncation error in an analytic form is given by **Taylor's Theorem**.

Truncation Error

Theorem 1 (**Taylor's Theorem**)

Suppose that $f \in C^n[a,b]$, $f^{(n+1)}$ *exists on* $[a,b]$, *and* $x_0 \in [a,b]$. *For every* $x \in [a,b]$, *there exists a number* ξ *between* x_0 *and* x *with*

$$f(x) = P_n(x) + R_n(x), \tag{4}$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots \\ &+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \end{aligned} \tag{5}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \tag{6}$$

Truncation Error

Here $P_n(x)$ is called the nth-degree Taylor polynomial for f about x_0, and $R_n(x)$ is called the remainder term (or truncation error) associated with $P_n(x)$.

- In fact, one of the common problems in numerical methods is to try to **determine a realistic bound for the truncation error** R_n when x is in some specified interval.

Example 2

Let $f(x) = \cos(x)$ and $x_0 = 0$. Determine

(i) The second-degree Taylor polynomial for f about x_0.

(ii) The second-degree Taylor polynomial at $x = 0.01$.

(iii) Error bound at $x = 0.01$.

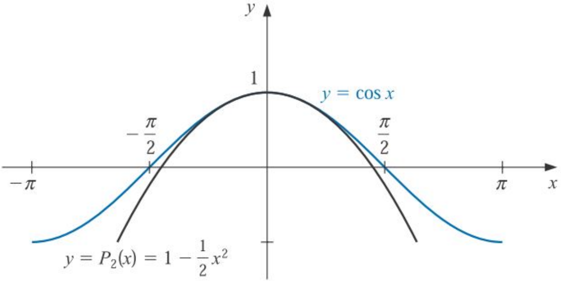
Truncation Error

Solution:

(i) $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, and $f'''(x) = \sin(x)$. For $n = 2$ and $x_0 = 0$, we have

$$\cos(x) \approx P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \quad (7a)$$

$$= 1 - \frac{1}{2}x^2. \quad (7b)$$



Truncation Error

(ii) When $x = 0.01$, Taylor polynomial becomes

$$P_2(0.01) = 1 - \frac{1}{2}(0.01)^2 = 0.99995. \tag{8}$$

(iii) $$R_2(x) = \frac{f'''(\xi)}{3!}x^3, \tag{9}$$

where ξ is some unknown number between 0 and x. Hence,

$$|R_2(0.01)| = \frac{1}{6}(0.01)^3 \left|\sin(\xi)\right| \le \frac{1}{6}(0.01)^3 = 0.1\bar{6} \times 10^{-6}. \tag{10}$$

Truncation Error

Remark 1.

$$E_a = |\cos(0.01) - P_2(0.01)| \approx |0.999950000416665 - 0.99995|$$
$$= 4.16665 \times 10^{-10} < |R_2(0.01)| = 0.1\bar{6} \times 10^{-6}.$$

So the actual absolute error is less than the upper bound as expected.

Remark 2.

The nth Taylor polynomial about the number x_0 is an **excellent approximation** to an $(n+1)$-times differentiable function f **in a small neighborhood of x_0**.