Chapter 2
Numerical Solution of Nonlinear Equations

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Outline

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Chapter Questions

In this chapter we shall try to answer the following main FIVE questions:

1. **HOW TO APPROXIMATE A ROOT OF A NONLINEAR EQUATION USING THE BISECTION METHOD?**
2. **HOW TO ESTIMATE THE ERROR OF THE BISECTION METHOD?**
3. **WHAT IS THE NUMBER OF ITERATIONS REQUIRED BY THE BISECTION METHOD TO ACHIEVE A CERTAIN TOLERANCE?**
4. **HOW TO APPROXIMATE A ROOT OF A NONLINEAR EQUATION USING NEWTON'S METHOD?**
5. **HOW TO APPROXIMATE THE rTH-ROOT OF A REAL NUMBER N USING NEWTON'S METHOD?**

Outline

Introduction

- One of the most basic problems of numerical approximation is the **root-finding problem**.
  - This process involves finding **a root**, or a solution, of an equation of the form $f(x) = 0$, for a given function f.
  - A root of this equation is also called **a zero of the function f**.
- The problem of finding an approximation to the root of an equation can be traced back at least to 1700 B.C.; in particular, to **the days of the Babylonians** who presented a method for approximating the value of $\sqrt{2}$.

Introduction

- One method to obtain an approximate solution is to **plot the function** and determine **where it crosses the *x*-axis**.
    - Although **graphical methods** are useful for obtaining rough estimates of roots, ***they are limited because of their lack of precision***.
- An alternative approach is to use **trial and error**, which involves **guessing a value of *x*** and evaluating whether $f(x)$ is zero or not. This process is then often repeated until a better estimate of the root is obtained.
    - Such haphazard method is obviously **inefficient** and **inadequate**; because it is always preferable to have methods that **come up with the approximate solution automatically**.

Introduction

- ***Numerical methods*** represent alternatives that are also approximate but ***employ systematic strategies*** to home in on the true root.
    - The combination of these systematic methods and computers makes the solution of most applied root-finding problems a **simple** and **efficient** task.
- Numerical methods for solving nonlinear equations can be divided into two main categories: **bracketing methods** and **open methods**.
    - In both classes of methods, algorithms perform best when they take advantage of *known characteristics* of the given function such as the **continuity** or the **differentiability** of the function.

Introduction

- **Bracketing methods** for finding roots start with **two guesses** that bracket the root, say p, and then **systematically reduce the width of the bracket**.
  - These methods are **globally convergent**[1], but their **convergence is usually very slow**.
- **Open methods** seek to find an approximate root iteratively; however, they generally require only a **single starting value** or **two starting values** that **do not necessarily bracket a root**.
  - These methods are usually ***more computationally efficient than bracketing methods***, but ***they do not always work for any initial approximation***, since they are **locally convergent**[2].

[1] They converge for an arbitrary initial interval that brackets a root.

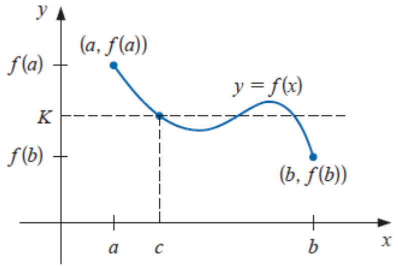
[2] An iterative method is called locally convergent if the successive approximations produced by the method are guaranteed to converge to a solution when the initial approximation is already ***close enough*** to the solution.

Outline

Preliminary Theorem and Corollary

Theorem 1 (**Intermediate Value Theorem**)

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$.

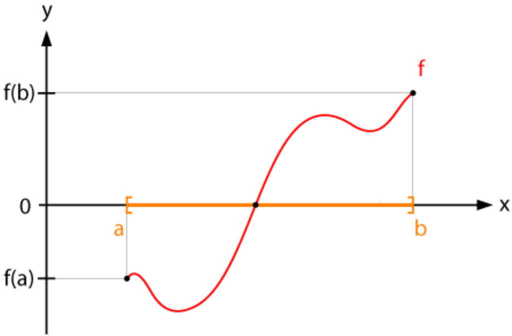


Preliminary Theorem and Corollary

The Intermediate Value Theorem leads to the following useful corollary:

Corollary 1

*Suppose $f \in C[a,b]$, with $f(a)$ and $f(b)$ of opposite sign, i.e. $f(a) \cdot f(b) < 0$. The Intermediate Value Theorem implies that a number (**root**) p exists in (a,b) with $f(p) = 0$.*



Outline

The Bisection Method

The first technique in the class of **bracketing methods** is called the **bisection**[3], or **binary-search**, **method**.

- It works when the function f is a **continuous function**.

[3] Bisection is the division of the interval into two equal parts (halves).

Algorithm Description

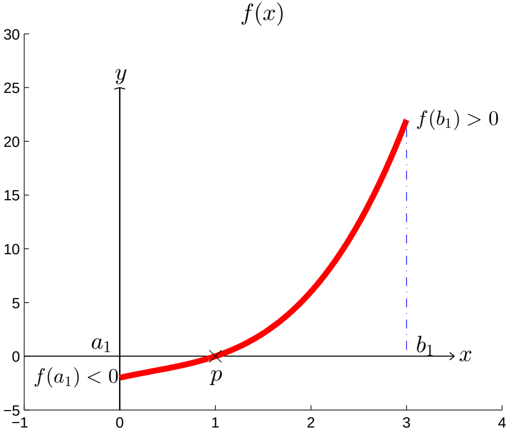
To begin, determine any two numbers a and b such that $f(a) \cdot f(b) < 0$, then set $a_1 = a$ and $b_1 = b$.

Let p_1 be the midpoint of $[a_1, b_1]$; that is, $p_1 = \frac{a_1 + b_1}{2}$.

- If $|f(p_1)| \leq \epsilon$, for some relatively small positive number ϵ, then $p \approx p_1$, and we are done.
- If $|f(p_1)| > \epsilon$, then $f(p_1)$ has an opposite sign with either $f(a_1)$ or $f(b_1)$.
  - If $f(p_1)$ and $f(a_1)$ have opposite signs, then $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.
  - If $f(p_1)$ and $f(a_1)$ have the same sign, then $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.

Then reapply the process to the interval $[a_2, b_2]$.

Illustrative Animation



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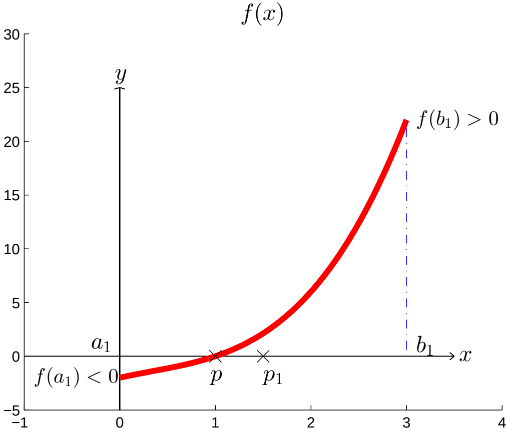

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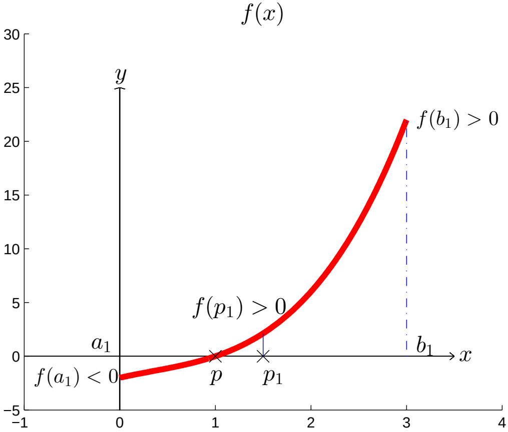

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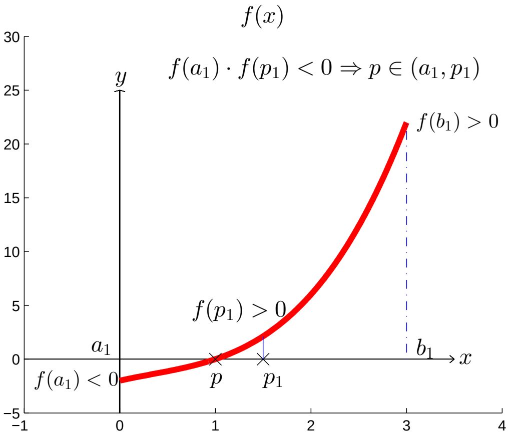

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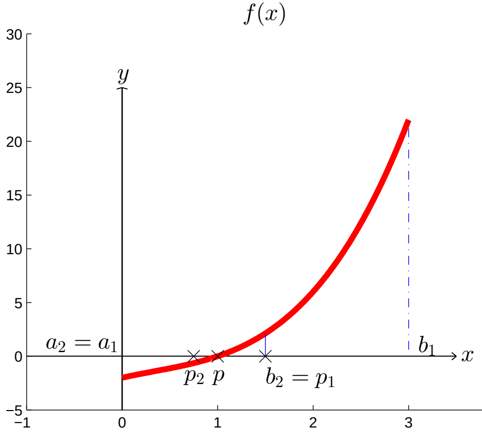

The Bisection Method

Stopping criteria

We select a tolerance $\epsilon > 0$, and generate approximations $p_1, \ldots, p_n$ until one of the following conditions (stopping criteria) is met:

$$|b_n - a_n| \leq \epsilon, \tag{1}$$

$$|f(p_n)| \leq \epsilon, \tag{2}$$

$$|p_n - p_{n-1}| \leq \epsilon, \tag{3}$$

$$\frac{|p_n - p_{n-1}|}{|p_n|} \leq \epsilon, \quad p_n \neq 0, \tag{4}$$

for some $n \in \mathbb{Z}^+$.

The Bisection Method

Example 1

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$. Then apply the bisection method to determine an approximation p_n to the root such that

$$|f(p_n)| \leq 10^{-2},$$

where p_i is the approximation obtained by the bisection method at iteration i.

The Bisection Method

Solution 1

Because $f(1) = -5$ and $f(2) = 14$, the Intermediate Value Theorem ensures that this continuous function has a root in $[a_1, b_1] = [1, 2]$.

- *For the first iteration of the bisection method, we use the fact that at the midpoint of $[1, 2]$, $p_1 = 1.5$, we have $f(p_1) = f(1.5) = 2.375 > 0$.*
    - *This indicates that we should select the interval $[a_2, b_2] = [1, 1.5]$ for our second iteration, since $f(1) \cdot f(1.5) < 0$.*
- *Then we find that $f(p_2) = f(1.25) = -1.796875$, so our new interval becomes $[a_3, b_3] = [1.25, 1.5]$, whose midpoint is $p_3 = 1.375$.*

The Bisection Method

Continuing in this manner, the bisection method generates approximations as shown in the following table.

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072

We find that after 9 iterations, $p_9 = 1.365234375$, satisfies

$$|f(p_9)| \approx 0.000072 < 10^{-2} \Rightarrow p \approx p_9 = 1.365234375.$$

Outline

Error Analysis

Theorem 2

Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$\boxed{|p - p_n| \le \frac{b-a}{2^n}}, \tag{5}$$

when $n \ge 1$.

Error Analysis

Proof.

Since $b_1 - a_1 = b - a$,

$$b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2}(b - a),$$

$$b_3 - a_3 = \frac{1}{2}(b_2 - a_2) = \frac{1}{2^2}(b_1 - a_1) = \frac{1}{2^2}(b - a),$$

$$\vdots$$

Then for each $n \geq 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b - a), \tag{6}$$

and $p \in (a_n, b_n)$. Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \geq 1$, it follows that

$$|p - p_n| \leq \frac{1}{2}(b_n - a_n) \stackrel{(6)}{=} \frac{b - a}{2^n}.$$

□

Outline

The Number of Iterations Required for a Certain Tolerance

Theorem 2 can be used to determine in advance ***the number of iterations n that the bisection method would need to converge to a root to within a certain tolerance*** ϵ.

From Inequality (5), if we set

$$|p - p_n| \leq \frac{b-a}{2^n} \leq \epsilon,$$

then taking the logarithms of both sides yields

$$n \geq \frac{\log\left(\dfrac{b-a}{\epsilon}\right)}{\log(2)}. \tag{7}$$

The Number of Iterations Required for a Certain Tolerance

Remark 1.

Logarithms to any base would suffice, but it is preferable to **use base-10 logarithms when the tolerance is given as a power of 10**.

The Number of Iterations Required for a Certain Tolerance

Example 2

Determine the number of iterations necessary to solve $f(x) = x^3 + 4\,x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Solution 2

From Inequality (7)*, we find that*

$$n \geq \frac{\log\left(\frac{2-1}{10^{-3}}\right)}{\log(2)} = \frac{3}{\log(2)} \approx 9.97. \tag{8}$$

$$\Rightarrow n = 10. \tag{9}$$

Hence, ten iterations will ensure an approximation accurate to within 10^{-3}.

Outline

Pros and Cons

Pros

1. The method has the important property that it **always converges** to a solution.

Cons

1. It is **relatively slow to converge**.
2. A **good intermediate approximation might be inadvertently discarded**.
3. If a function $f(x)$ is such that it just **touches the *x*-axis** such as $f(x) = x^2 = 0$, it will be unable to find the lower guess a and the upper guess b such that $f(a) \cdot f(b) < 0$.