Chapter 2
Numerical Solution of Nonlinear Equations

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26 October, 2015

Outline

Newton's Method

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Newton's Method

Newton's method (also known as the **Newton-Raphson method**), named after *Isaac Newton* and *Joseph Raphson*, is one of the most powerful and well-known numerical methods for solving a root-finding problem that belongs to the class of **open methods**.

- It works when the function f have a **continuous derivative** in the neighborhood of the root.

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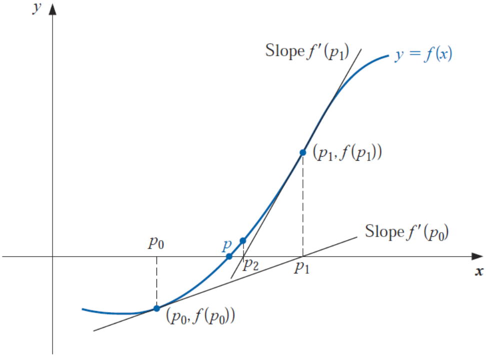
Newton's Method
Graphical Derivation
Derivation Based on Taylor Polynomials
Pros and Cons
Approximating the rth-Root of a Real Number N

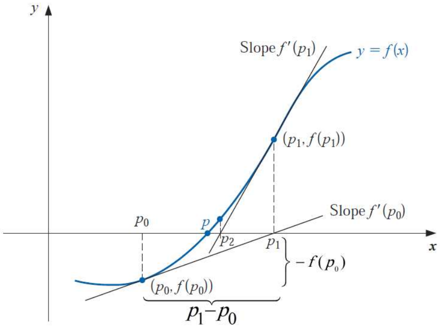
Graphical Derivation

The figure illustrates how the approximations are obtained using successive tangents. Starting with the initial approximation p_0, the approximation p_1 is the ***x*-intercept of the tangent line** to the graph of f at $(p_0, f(p_0))$.



Graphical Derivation

Since $f'(p_0) = \dfrac{-f(p_0)}{p_1 - p_0} \Rightarrow p_1 = p_0 - \dfrac{f(p_0)}{f'(p_0)}$.



Graphical Derivation

The approximation p_2 is the x-intercept of the tangent line to the graph of f at $(p_1, f(p_1))$. Hence

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}.$$

Repeatedly, Newton's method generates a sequence of approximations $\{p_n\}_{n=0}^{\infty}$, by letting

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1, \tag{1}$$

which is called **Newton-Raphson's formula.**

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Derivation Based on Taylor Polynomials

Suppose that $f \in C^2[a,b]$, and let $p_0 \in [a,b]$ be a sufficiently close approximation to the root p such that $f'(p_0) \neq 0$. Consider the first Taylor polynomial for $f(x)$ expanded about p_0:

$$f(x) = f(p_0) + (x - p_0)f'(p_0) + \frac{(x - p_0)^2}{2} f''(\xi), \tag{2}$$

where ξ lies between x and p_0. At $x = p$, this equation gives

$$f(p) = 0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi). \tag{3}$$

Since $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0). \tag{4}$$

Derivation Based on Taylor Polynomials

Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} = p_1. \tag{5}$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1}.$$

Newton's Method

Example 1

Consider the function $f(x) = \cos(x) - x = 0$. Approximate a root of f using Newton's method with initial guess $\pi/4$.

Newton's Method

Solution 1

To apply Newton's method to this problem we need $f'(x) = -\sin(x) - 1$. Starting with $p_0 = \pi/4$, we generate the sequence defined, for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{\cos(p_{n-1}) - p_{n-1}}{-\sin(p_{n-1}) - 1}. \tag{6}$$

Therefore,

$$p_1 = p_0 - \frac{\cos(p_0) - p_0}{-\sin(p_0) - 1} \approx 0.7395361337, \tag{7}$$

$$p_2 = p_1 - \frac{\cos(p_1) - p_1}{-\sin(p_1) - 1} \approx 0.7390851781, \tag{8}$$

$$\vdots$$

Newton's Method

Solution 1

This gives the approximations in the table below. An excellent approximation is obtained with $n = 3$. Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

Newton's Method

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

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Pros and Cons

Pros

1. In general, Netwon's method has **quadratic convergence**: the error is squared at each iteration (**the number of accurate digits doubles**).

Cons

1. Newton's method **requires calculating the derivative**; however, an analytical expression for the derivative may not be easy to calculate.
2. If the initial value is not sufficiently close from the true zero, Newton's method **may fail to converge** (**has only local convergence**).

Pros and Cons

Cons

3. For some functions and some starting points, **it may enter an infinite cycle**, preventing convergence.
    - Check the function $f(x) = x^3 - 2x + 2$ with 0 as the starting point.

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Approximating the rth-Root of a Real Number N

Newton's method is very suitable for **computing approximate values of higher rth-roots of positive numbers.** The value of $\sqrt[r]{N}$ is the zero of the function $f(x) = x^r - N$. Newton's formula

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad n \geq 0,$$

applied to the function f reads

$$p_{n+1} = p_n - \frac{p_n^r - N}{r\, p_n^{r-1}} = \frac{1}{r}\left(r p_n - \frac{p_n^r - N}{p_n^{r-1}}\right). \tag{9}$$

$$\Rightarrow \boxed{p_{n+1} = \frac{1}{r}\left((r-1)p_n + \frac{N}{p_n^{r-1}}\right), \quad n \geq 0.} \tag{10}$$

Approximating the rth-Root of a Real Number N

Example 2

Calculate $\sqrt{19}$ rounded to 5 decimal digits.

Solution 2

Since $\sqrt{16} < \sqrt{19} < \sqrt{25} \Rightarrow 4 < \sqrt{19} < 5$. *Hence we can choose* $p_0 = 4.5$ *as an initial estimate. Substituting* $r = 2$ *and* $N = 19$ *in* (10) *yields*

$$p_1 = \frac{1}{2}\left(p_0 + \frac{N}{p_0}\right) = \frac{1}{2}\left(4.5 + \frac{19}{4.5}\right) \approx 4.36111.$$

$$p_2 = \frac{1}{2}\left(p_1 + \frac{N}{p_1}\right) \approx \frac{1}{2}\left(4.36111 + \frac{19}{4.36111}\right) \approx 4.35890.$$

Approximating the *r*th-Root of a Real Number *N*

Solution 2

$$p_3 = \frac{1}{2}\left(p_2 + \frac{N}{p_2}\right) \approx \frac{1}{2}\left(4.35890 + \frac{19}{4.35890}\right) \approx 4.35890.$$

Hence $\sqrt{19} \approx 4.35890$ *rounded to* 5 *decimal digits.*