Chapter 3
Curve Fitting: Interpolation

Dr. Kareem Elgindy

Lecturer
Mathematics Department, Faculty of Science
Assiut University

02 November, 2015

Outline

Outline

Interpolation

- Suppose that a census of the population of some country is taken every 10 years from, say 1950 to 2000. One may ask if we could use this tabulated data to obtain a reasonable estimate of the population, say, in 1975 or even in the future, say in year 2020.

- Predictions of this type can be obtained by using a **function** that fits the given data.
  - If the prediction is required at an intermediate year/input **inside** the tabulated or observed range, then this process is called **interpolation**[1].
  - If the prediction is required at a year **outside** the tabulated or observed range, then this process is called **extrapolation**[2].

[1] The prefix "inter" means "in between" or "among".
[2] The prefix "extra" means "outside" or "in addition to".

Interpolation

- Interpolation is ***far more likely*** to give a reliable result than extrapolation.
  - The reason is that extrapolation represents a step into the unknown because the process extends the approximate curve beyond the known region. As such, **the true curve could easily diverge from the prediction**.
- In this chapter, we shall focus our study on **interpolation** only.

Definition 1 (Interpolation)

Interpolation is the estimation of intermediate values between precise data points. In other words, **interpolation is the determination/estimation of a function of x, $f(x)$, from certain known values of the function.**

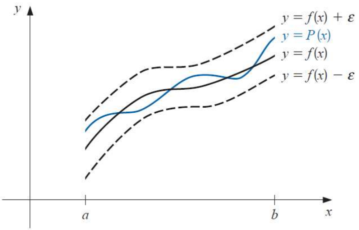
- In particular, if $x_0 < \ldots < x_n$ and $y_0 = f(x_0), \ldots, y_n = f(x_n)$ are known, and if $x_0 < x < x_n : x \neq x_i\, \forall i$, then the estimation of $f(x)$ is said to be an interpolation.

Interpolation

- If the function that fits the given data is a **polynomial**, $P(x)$, then the process is called **polynomial interpolation** and is the subject of this chapter.
  - By definition,

$$\boxed{P(x_i) = f(x_i), \quad i = 0, \ldots, n}. \tag{1}$$

- One reason for the importance of polynomials is that **they uniformly approximate continuous functions**.
  - By this we mean that given any function, defined and continuous on a closed and bounded interval, **there exists a polynomial that is as "close" to the given function as desired**.

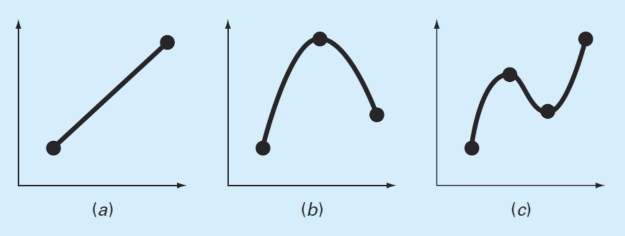


Interpolation

Remark 1.

For n data points, there is **one and only one** polynomial of order $(n-1)$ that passes through all the points.

For example, there is only one straight line (i.e., a first-order polynomial) that connects two points. Similarly, only one parabola connects a set of three points, and only one third-order (cubic) polynomial connecting four points, etc.



Outline

Finite-Differences

- The Interpolation process depends upon the **finite-difference** concept. Consider a set of equally-spaced nodes $\{x_i\}_{i=0}^{n}, n \in \mathbb{Z}^+$; i.e. $x_i - x_{i-1} =$ constant $= h \in \mathbb{R}\ \forall i$. Let also $f_0 = f(x_0), f_1 = f(x_1), \ldots, f_n = f(x_n)$, be their corresponding values for some function f. Then $\boldsymbol{f_1 - f_0, f_2 - f_1, f_3 - f_2, \ldots, f_n - f_{n-1}}$ **are called finite-differences**.
- To find an estimate for $f(x)$, for any value of x, we use some interpolation formulas that can be derived using the three useful operators: the **forward-difference operator**, the **backward-difference operator**, and the **shift operator**.
  - *The former two operators are defined using the finite-differences.*

Outline

Forward-Difference Operator

The difference of the function values gives us the first forward-difference, and is denoted by Δ.

Definition 2 (**First Forward-Difference Operator**)

The first forward-difference operator of a function f w.r.t. x_i is defined as

$$\Delta f_i = f_{i+1} - f_i \ \forall i \geq 0. \tag{2}$$

The difference of the first forward-differences gives us the second forward-difference denoted by Δ^2.

Definition 3 (**Second Forward-Difference Operator**)

The second forward-difference operator of a function f w.r.t. x_i is defined as

$$\Delta^2 f_i = \Delta(\Delta f_i) \overset{(2)}{=} \Delta f_{i+1} - \Delta f_i = f_{i+2} - 2f_{i+1} + f_i \ \forall i \geq 0.$$

Forward-Difference Operator

Definition 4 (*k*th Forward-Difference Operator)

In general, the kth forward-difference operator of f w.r.t. x_i is defined recursively as

$$\Delta^k f_i = \Delta(\Delta^{k-1} f_i) = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i \ \forall i \geq 0, k \geq 2.$$

The determination of the forward-differences from tabulated data points is outlined in the following **forward-difference table** using four data points $\{(x_i, f_i)\}_{i=0}^{3}$.

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	x_0	f_0			
			$\Delta f_0 = f_1 - f_0$		
1	x_1	f_1		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	
			$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
2	x_2	f_2		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	
			$\Delta f_2 = f_3 - f_2$		
3	x_3	f_3			

Outline

Backward-Difference Operator

Definition 5 (**First Backward-Difference Operator**)

The first backward-difference operator of a function f w.r.t. x_i is defined as

$$\nabla f_i = f_i - f_{i-1} \; \forall i \geq 1. \tag{3}$$

Definition 6 (**Second Backward-Difference Operator**)

The second backward-difference operator of a function f w.r.t. x_i is defined as

$$\nabla^2 f_i = \nabla(\nabla f_i) \overset{(3)}{=} \nabla f_i - \nabla f_{i-1} = f_i - 2f_{i-1} + f_{i-2} \; \forall i \geq 2.$$

Definition 7 (***k*th Backward-Difference Operator**)

In general, the kth backward-difference operator of f w.r.t. x_i is defined recursively as

$$\nabla^k f_i = \nabla(\nabla^{k-1} f_i) = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1} \; \forall i \geq k, k \geq 2.$$

Backward-Difference Operator

The determination of the backward-differences from tabulated data points is outlined in the following **backward-difference table** using four data points $\{(x_i, f_i)\}_{i=0}^{3}$.

i	x_i	f_i	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$
0	x_0	f_0			
			$\nabla f_1 = f_1 - f_0$		
1	x_1	f_1		$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	
			$\nabla f_2 = f_2 - f_1$		$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$
2	x_2	f_2		$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$	
			$\nabla f_3 = f_3 - f_2$		
3	x_3	f_3			

Outline

Shift Operator

Definition 8 (**Shift Operator**)

Consider a set of data points $\{(x_i, f(x_i))\}_{i=0}^n, n \in \mathbb{Z}^+$, where $\{x_i\}_{i=0}^n$ is a set of equally-spaced nodes with a common difference h.

- The shift operator E is an operator that takes the function value $f(x_i)$ to its translation $f(x_i + h) = f(x_{i+1})$, that is

$$Ef(x_i) = f(x_i + h) = f(x_{i+1}) \ \forall i. \tag{4}$$

For simplicity, if we denote $f(x_i)$ by $f_i \ \forall i$, then the shift operator E is defined as

$$Ef_i = f_{i+1} \ \forall i. \tag{5}$$

Clearly,

$$E^2 f_i = E(Ef_i) \overset{(5)}{=} E(f_{i+1}) = f_{i+2} \ \forall i. \tag{6}$$

In general,

$$E^k f_i = E(E^{k-1} f_i) = \ldots = f_{i+k} \ \forall i, k. \tag{7}$$

Shift Operator

Relationship between the forward-difference operator and the shift operator

Since $\Delta f_i = f_{i+1} - f_i = E\, f_i - f_i = (E-1) f_i$, then

$$\boxed{\Delta = E - 1}.$$

Relationship between the backward-difference operator and the shift operator

Since $\nabla f_i = f_i - f_{i-1} = f_i - E^{-1}\, f_i = (1 - E^{-1}) f_i$, then

$$\boxed{\nabla = 1 - E^{-1} \text{ or } E = (1-\nabla)^{-1}}.$$

Outline

Gregory-Newton Forward-Difference Interpolation Formula

Gregory-Newton's forward-difference formula is a **finite difference identity** giving an interpolated value $f(x)$ between tabulated points $\{f_i\}_{i=0}^{n}, n \in \mathbb{Z}^+$, **in terms of the first value f_0** and **the powers of the forward-difference operator Δ**. The formula was named after *Isaac Newton* and *James Gregory*.

Let $\{(x_i, f_i)\}_{i=0}^{n}$, be a given tabulated data, where $\{x_i\}_{i=0}^{n}$ are equally-spaced nodes with a common difference h; i.e. $x_i = x_0 + i\,h \ \forall i$. To estimate the value of f at a certain value $x_s = x_0 + s\,h, 0 < s < 1$, denote $f(x_s)$ by f_s; i.e. $f_s = f(x_0 + s\,h)$.

$$\because f_s = f(x_0 + s\,h) = E^s f_0 = (1 + \Delta)^s f_0.$$

$$\Rightarrow f_s = \left(1 + \frac{s}{1!}\Delta + \frac{s(s-1)}{2!}\Delta^2 + \frac{s(s-1)(s-2)}{3!}\Delta^3 + \ldots\right) f_0.$$

$$\Rightarrow f_s = f_0 + \frac{s}{1!}\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots.$$

Gregory-Newton Forward-Difference Interpolation Formula

$$\therefore f_s \approx P_n(s) = f_0 + \frac{s}{1!}\Delta f_0 + \ldots + \frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n f_0, \; n \geq 0 \tag{8}$$

Equation (8) is known as the **Gregory-Newton forward-difference interpolation formula**, and $P_n(s)$ is called the **nth-degree Gregory-Newton forward-difference interpolating polynomial**.