Chapter 3
Curve Fitting: Interpolation

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Outline

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Gregory-Newton Forward-Difference Interpolation Formula

Remark 1.

Using the **binomial coefficient** notation $\left(\begin{array}{c} n \\ k \end{array}\right)$ defined by

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{(n-k)!\,k!} = \frac{n\,(n-1)\,(n-2)\ldots(n-k+1)}{k!}, \tag{1}$$

we can rewrite Gregory-Newton forward-difference interpolation formula in a **compact form** as follows:

$$f_s \approx P_n(s) = f_0 + \left(\begin{array}{c} s \\ 1 \end{array}\right)\Delta f_0 + \left(\begin{array}{c} s \\ 2 \end{array}\right)\Delta^2 f_0 + \ldots + \left(\begin{array}{c} s \\ n \end{array}\right)\Delta^n f_0,\ n \geq 0$$

$$= \boxed{f_0 + \sum_{k=1}^{n} \left(\begin{array}{c} s \\ k \end{array}\right)\Delta^k f_0, \quad n \geq 0.} \tag{2}$$

Gregory-Newton Forward-Difference Interpolation Formula

Remark 2.

Gregory-Newton forward-difference interpolation formula is **most suitable when the interpolation is required near the beginning of the table**; that is, **when x_s is near x_0.**

Gregory-Newton Forward-Difference Interpolation Formula

Required steps for interpolating a function f at a certain point x_s such that $x_0 < x_s < x_1$:

1. Compute the parameter s from the formula $s = \frac{x_s - x_0}{h}$.
2. Construct the forward-difference table.
3. Apply Gregory-Newton forward-difference interpolation formula.

Gregory-Newton Forward-Difference Interpolation Formula

Example 1

Consider the table of data given below. Use Gregory-Newton forward-difference interpolation formula to approximate $f(1.25)$.

x	1.0	1.5	2.0	2.5
$f(x)$	4.0	18.25	44.00	84.25

Solution 1

Let $x_0 = 1.0, h = 0.5$, *and* $x_s = 1.25$. *Since* $x_s = x_0 + s\,h$, *then*

$$s = \frac{x_s - x_0}{h} = \frac{1.25 - 1.0}{0.5} = 0.5. \tag{3}$$

Gregory-Newton Forward-Difference Interpolation Formula

Solution 1

The forward-difference table can be constructed as follows:

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	1.0	4.0			
			14.25		
1	1.5	18.25		11.5	
			25.75		3.0
2	2.0	44.00		14.5	
			40.25		
3	2.5	84.25			

From the table, we have $f_0 = 4.0, \Delta f_0 = 14.25, \Delta^2 f_0 = 11.5$, *and* $\Delta^3 f_0 = 3.0$.

Gregory-Newton Forward-Difference Interpolation Formula

Solution 1

Gregory-Newton forward-difference interpolation formula is given by

$$\begin{aligned} f_s \approx P_3(s) &= f_0 + \sum_{k=1}^{3} \binom{s}{k} \Delta^k f_0 \\ &= f_0 + \frac{s}{1!}\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0. \end{aligned}$$

$$\begin{aligned} \Rightarrow f(1.25) &\approx 4.0 + 0.5(14.25) + \frac{(0.5)(-0.5)}{2!}(11.5) \\ &+ \frac{(0.5)(-0.5)(-1.5)}{3!}(3.0) = 9.875. \end{aligned}$$

Outline

Gregory-Newton Backward-Difference Interpolation Formula

Gregory-Newton's backward-difference formula is a **finite difference identity** giving an interpolated value $f(x)$ between tabulated points $\{f_i\}_{i=0}^{n}, n \in \mathbb{Z}^+$, **in terms of the last value f_n** and **the powers of the backward-difference operator ∇**.

Let $\{(x_i, f_i)\}_{i=0}^{n}$, be a given tabulated data, where $\{x_i\}_{i=0}^{n}$ are equally-spaced nodes with a common difference h; i.e. $x_i = x_0 + i\,h\ \forall i$. To estimate the value of f at a certain value $x_s = x_n + s\,h, -1 < s < 0$, denote $f(x_s)$ by f_s; i.e. $f_s = f(x_n + s\,h)$.

$$\because f_s = f(x_n + s\,h) = E^s f_n = (1-\nabla)^{-s} f_n.$$

$$\Rightarrow f_s = \left(1 + \frac{s}{1!}\nabla + \frac{s(s+1)}{2!}\nabla^2 + \frac{s(s+1)(s+2)}{3!}\nabla^3 + \ldots\right) f_n.$$

$$\Rightarrow f_s = f_n + \frac{s}{1!}\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \ldots . \tag{4}$$

Gregory-Newton Backward-Difference Interpolation Formula

$$\therefore f_s \approx P_n(s) = f_n + \frac{s}{1!}\nabla f_n + \ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f_n, n \geq 0. \quad (5)$$

Series (4) is known as the **Gregory-Newton backward-difference interpolation formula**, and $P_n(s)$ is called the **nth-degree Gregory-Newton backward-difference interpolating polynomial**.

Gregory-Newton Backward-Difference Interpolation Formula

Remark 3.

Using the **binomial coefficient** notation, we can rewrite Gregory-Newton backward-difference interpolation formula in a **compact form** as follows: since

$$\binom{-s}{k} = \frac{-s(-s-1)\ldots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\ldots(s+k-1)}{k!}.$$

$$\Rightarrow \frac{s(s+1)\ldots(s+k-1)}{k!} = (-1)^k \binom{-s}{k}.$$

$$\therefore f_s \approx P_n(s) = f_n + \frac{s}{1!}\nabla f_n + \ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f_n$$

$$= \boxed{f_n + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f_n, \quad n \geq 0.} \tag{6}$$

Gregory-Newton Backward-Difference Interpolation Formula

Remark 4.

Gregory-Newton backward-difference interpolation formula is **most suitable when the interpolation is required near the end of the table**; that is, **when x_s is near x_n.**

Gregory-Newton Backward-Difference Interpolation Formula

Required steps for interpolating a function f at a certain point x_s such that $x_{n-1} < x_s < x_n$:

1. Compute the parameter s from the formula $s = \frac{x_s - x_n}{h}$.
2. Construct the backward-difference table.
3. Apply Gregory-Newton backward-difference interpolation formula.

Gregory-Newton Backward-Difference Interpolation Formula

Example 2

Consider the table of data given below. Use Gregory-Newton backward-difference interpolation formula to approximate $f(1.457)$.

x	1.0	1.1	1.2	1.3	1.4	1.5
$f(x)$	1	1.331	1.728	2.147	2.744	3.375

Solution 2

Let $x_n = x_5 = 1.5, h = 0.1$*, and* $x_s = 1.457$*. Since* $x_s = x_n + s\,h$*, then*

$$s = \frac{x_s - x_n}{h} = \frac{1.457 - 1.5}{0.1} = -0.43. \tag{7}$$

Gregory-Newton Backward-Difference Interpolation Formula

Solution 2

The backward-difference table can be constructed as follows:

i	x_i	f_i	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$	$\nabla^4 f_i$	$\nabla^5 f_i$
0	1.0	1					
			0.331				
1	1.1	1.331		0.066			
			0.397		−0.044		
2	1.2	1.728		0.022		0.2	
			0.419		0.156		−0.5
3	1.3	2.147		0.178		−0.3	
			0.597		−0.144		
4	1.4	2.744		0.034			
			0.631				
5	1.5	3.375					

From the table, we have $f_n = f_5 = 3.375, \nabla f_5 = 0.631, \nabla^2 f_5 = 0.034, \nabla^3 f_5 = -0.144,$ $\nabla^4 f_5 = -0.3$, *and* $\nabla^5 f_5 = -0.5$.

Gregory-Newton Backward-Difference Interpolation Formula

Solution 2

Gregory-Newton backward-difference interpolation formula is given by

$$
\begin{aligned}
f_s \approx P_5(s) &= f_n + \sum_{k=1}^{5} (-1)^k \binom{-s}{k} \nabla^k f_n \\
&= f_5 + \frac{s}{1!}\nabla f_5 + \frac{s(s+1)}{2!}\nabla^2 f_5 + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_5 \\
&+ \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_5 + \frac{s(s+1)(s+2)(s+3)(s+4)}{5!}\nabla^5 f_5.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow f(1.457) &\approx 3.375 + (-0.43)(0.631) + \frac{(-0.43)(0.57)}{2!}(0.034) \\
&+ \frac{(-0.43)(0.57)(1.57)}{3!}(-0.144) \\
&+ \frac{(-0.43)(0.57)(1.57)(2.57)}{4!}(-0.3) \\
&+ \frac{(-0.43)(0.57)(1.57)(2.57)(3.57)}{5!}(-0.5) \approx 3.135811,
\end{aligned}
$$

rounded to 6 *decimal digits.*

Outline

Lagrange Interpolating Polynomials

Lagrange polynomials are used for polynomial interpolation **for both equally- and unequally- spaced nodes $x_i, i = 0, \ldots, n$**. Although named after *Joseph Louis Lagrange*, who published it in 1795, it was first discovered in 1779 by *Edward Waring* and it is also an easy consequence of a formula published in 1783 by *Leonhard Euler*.

Consider the problem of determining a polynomial of degree one (linear interpolating polynomial) that passes through the distinct points (x_0, f_0) and (x_1, f_1). Define the Lagrange coefficient polynomials

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

Then $L_0(x_0) = 1$, $L_0(x_1) = 0$, $L_1(x_0) = 0$, and $L_1(x_1) = 1$. The **linear Lagrange interpolating polynomial** through (x_0, f_0) and (x_1, f_1) is

$$\boxed{P_1(x) = L_0(x) f_0 + L_1(x) f_1.} \tag{8}$$

Note that

$$\left.\begin{aligned} P_1(x_0) &= 1 \cdot f_0 + 0 \cdot f_1 = f_0, \\ P_1(x_1) &= 0 \cdot f_0 + 1 \cdot f_1 = f_1. \end{aligned}\right\} \text{ (\textit{interpolation conditions})}$$

Lagrange Interpolating Polynomials

The same strategy can be employed to fit a parabola through three points. Such a **second-order Lagrange interpolating polynomial** can be written as

$$\begin{aligned} P_2(x) &= L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 \\ &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f_2. \end{aligned} \tag{9}$$

Note that

$$\left.\begin{aligned} P_2(x_0) &= 1\cdot f_0 + 0\cdot f_1 + 0\cdot f_2 = f_0, \\ P_2(x_1) &= 0\cdot f_0 + 1\cdot f_1 + 0\cdot f_2 = f_1; \\ P_2(x_2) &= 0\cdot f_0 + 0\cdot f_1 + 1\cdot f_2 = f_2; \end{aligned}\right\}$$ (***interpolation conditions***)

Lagrange Interpolating Polynomials

For simplicity, we can write

$$P_2(x) = \sum_{k=0}^{2} L_k(x) f_k, \tag{10}$$

where

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2} \frac{(x - x_i)}{(x_k - x_i)}, \quad k = 0, 1, 2. \tag{11}$$

The symbol $\prod$ is used to write products compactly and parallels the symbol $\sum$, which is used for writing sums.

Lagrange Interpolating Polynomials

Theorem 1 (**Lagrange Interpolation Formula**)

If $x_0, x_1, \ldots, x_n$ *are* $n+1$ *distinct numbers and* f *is a function whose values are given at these numbers, then there exists a unique polynomial* $P_n(x)$ *of degree* n *with* $f_k = P_n(x_k),$ *for each* $k = 0, 1, \ldots, n$. *This polynomial is given by*

$$P_n(x) = \sum_{k=0}^{n} L_k(x) f_k, \tag{12}$$

where

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}, \tag{13}$$

for each $k = 0, 1, \ldots, n$.

Lagrange Interpolating Polynomials

Example 3

Evaluate $\sin 20^o 18'$ given $\sin 20^o = 0.34202$ and $\sin 21^o = 0.35837$.

Solution 3

The linear Lagrange interpolating polynomial through $(20, 0.34202)$ *and* $(21, 0.35837)$ *is*

$$\begin{aligned} P_1(x) &= L_0(x) f_0 + L_1(x) f_1 \\ &= \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1. \end{aligned}$$

To state $x = 20^o 18'$ *as an angle using common decimal notation, we write*

$$x = 20^o 18' = 20 + \frac{18}{60} = 20.3.$$

Lagrange Interpolating Polynomials

Solution 3

Hence

$$\sin 20^o 18' \approx P_1(20.3) = \frac{20.3 - 21}{20 - 21}(0.34202) + \frac{20.3 - 20}{21 - 20}(0.35837)$$
$$= 0.346925.$$