Chapter 6
Numerical Integration

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Outline

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Introduction

- Mathematically, the definite integration of a function $f(x)$ over an interval $[a, b] \subset \mathbb{R}$ is represented by $\int_a^b f(x)\,dx$, where f is called the **integrand** and the interval $[a, b]$ is the **interval of integration**. Also a and b are called the **lower** and **upper limits of integration**, respectively.
- For an integrable function f that takes both positive and negative values, the definite integral corresponds to the area under the positive part of the graph of f minus the area above the graph of the negative part of the graph.
  - For this reason we say that $\int_a^b f(x)\,dx$ is the **net signed area** between the graph of the function and the interval $[a, b]$.
- Definite integrals appear in many applications such as calculating the mean value of a continuous function, the length of a plane curve, the area between two curves, volumes, work, displacement, etc.

Introduction

- The need to approximate a definite integral often arises when the integrand function **has no explicit antiderivative** or when **its antiderivative is not easy to obtain**, as is typically the case in more realistic examples.
- In addition, **the underlying function is often unknown and defined only by measurement at discrete points**.

Definition 1 (Numerical Integration)

Numerical integration is the approximate computation of an integral using numerical techniques.

Numerical Integration

The basic formula used to approximate $\int_a^b f(x)\,dx$ is called **numerical quadrature**[1]. It uses a sum $\sum_{i=0}^{n} a_i f(x_i)$ to approximate $\int_a^b f(x)\,dx$, for some $\{x_i\}_{i=0}^{n} \subset [a,b]$ and $\{a_i\}_{i=0}^{n} \subset \mathbb{R}$; i.e., typically we write,

$$\int_a^b f(x)\,dx \approx \underbrace{\sum_{i=0}^{n} a_i f(x_i)}_{\text{numerical quadrature}} \quad .$$

- The methods of quadrature in this chapter are based on the **interpolating polynomials** studied earlier in Chapter 3.

[1] Quadrature is a historical term which means determining area.

Outline

1. Introduction
2. Newton-Cotes Formulas

Newton-Cotes Formulas

The **Newton-Cotes formulas** are the most common numerical integration schemes. They are based on the strategy of ***replacing a complicated function or tabulated data with a polynomial that is easy to integrate***:

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_N(x)\,dx,$$

where $P_N(x)$ is a polynomial of degree $N \in \mathbb{Z}^+$. For simplicity, let $\{x_i\}_{i=0}^n$ be a set of equally-spaced nodes with a common difference h such that $a = x_0 < x_1 < \ldots < x_n = b$, and $x_i = x_0 + i\,h\ \forall i$. Also, let $x = x_0 + s\,h$, for some real parameter s. Then

$$dx = h\,ds,\ \ s = 0 \text{ when } x = x_0;\ \ s = n \text{ when } x = x_n.$$

Hence

$$I_n = \int_a^b f(x)\,dx \approx \int_a^b P_N(x)\,dx = \int_{x_0}^{x_n} P_N(x)\,dx$$

Newton-Cotes Formulas

$$\Rightarrow I_n \approx h \int_0^n P_N(s)\, ds, \tag{1}$$

where I_n denotes the integration over the interval $[x_0, x_n]$. Eq. (1) is known as the **Newton-Cotes formula**. $P_N(s)$ can be obtained from **Newton-Gregory forward-difference interpolation formula** defined by,

$$\begin{aligned} P_N(s) &= f_0 + \binom{s}{1} \Delta f_0 + \binom{s}{2} \Delta^2 f_0 + \ldots + \binom{s}{N} \Delta^N f_0 \\ &= \sum_{k=0}^{N} \binom{s}{k} \Delta^k f_0. \end{aligned}$$

Newton-Cotes Formulas

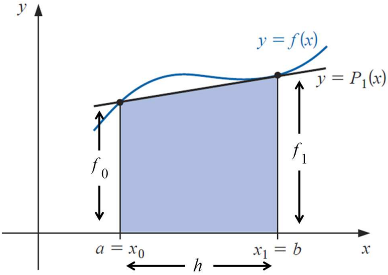
Hence, Newton-Cotes formula can be written in the form

$$I_n \approx h \int_0^n P_N(s)\, ds = h \int\limits_0^n \sum_{k=0}^{N} \binom{s}{k} \Delta^k f_0\, ds. \qquad (2)$$

Outline

Trapezoidal Rule

The **Trapezoidal rule** (also known as the **trapezoid rule** or **trapezium rule**) is a method for approximating a definite integral $\int_a^b f(x)dx$ using **linear approximations** of f. When f is a function with positive values, $\int_a^b f(x)\,dx$ is approximated by *the area in a trapezoid*.



Trapezoidal Rule

Hence if $f(x) \in C[x_0, x_1]$, the definite integral $\int_{x_0}^{x_1} f(x)dx$ can be approximated by the area of a trapezoid as follows:

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0 + f_1), \tag{3}$$

where $f_0 = f(x_0)$ and $f_1 = f(x_1)$ are the lengths of the parallel sides, and h is the height.

Derivation of the Trapezoidal Rule

Let $n = N = 1$, then $P_1(s) = f_0 + s\,\Delta f_0$. Therefore,

$$I_1 \approx h \int_0^1 P_1(s)\,ds = h \int_0^1 (f_0 + s\,\Delta f_0)\,ds.$$

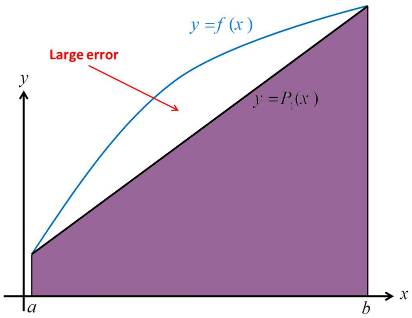
$$\Rightarrow I_1 \approx h \left[s\,f_0 + \frac{s^2}{2}\,\Delta f_0 \right]_0^1 = \frac{h}{2}(2f_0 + \Delta f_0),$$

where I_1 denotes the integration on the interval $[x_0, x_1]$. Since $\Delta f_0 = f_1 - f_0$, then

$$I_1 \approx \frac{h}{2}(f_0 + f_1). \tag{4}$$

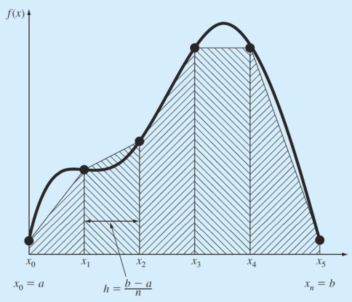
Disadvantage

- The Newton-Cotes formulas are generally **unsuitable for use over large integration intervals.** High-degree formulas would be required, but the values of the coefficients in these formulas are difficult to obtain.



How Can We Improve the Accuracy of the Trapezoidal Rule?

One way to improve the accuracy of the trapezoidal rule is to **divide the integration interval** from a to b into a number of segments and apply the method to each segment. The areas of individual segments can then be added to yield the integral for the entire interval. This technique is usually known as the **Composite Trapezoidal rule**.



Composite Trapezoidal Rule

Let $n > 1$. Then we have

$$\begin{aligned} I_n = \int_{x_0}^{x_n} f(x)\,dx &= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)\,dx \\ &\approx \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \ldots + \frac{h}{2}(f_{n-1} + f_n) \\ &= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n\right), \end{aligned}$$

which can be written in the general form

$$I_n \approx \frac{h}{2}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n\right). \tag{5}$$

Trapezoidal Rule

Example 1

Use the Trapezoidal rule with step-size $h = 0.2$ to calculate an approximate value to $\int_0^1 e^{x^2}\, dx$, rounded to 4 decimal places.

Solution 1

We construct the following table for the function $f(x) = e^{x^2}$ rounded to 4 decimal places.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.0000	1.0408	1.1735	2.4333	2.8965	2.7183

Trapezoidal Rule

Solution 1

$$\therefore \int_0^1 e^{x^2} dx \approx \frac{0.2}{2} \left[(1.0000 + 2.7183) + 2\,(1.0408 + 1.1735 + 1.4333 \right.$$
$$\left. +1.8965)\right] \approx 1.4807.$$