Chapter 7
Initial Value Problems

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Outline

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Introduction

A differential equation is an equation involving derivatives of a function or functions. If the function involves one independent variable, the equation is called an **ordinary differential equation (ODE)**; for instance,

$$f\left(t, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{1}$$

is an n**th-order ODE**. This is in contrast to a **partial differential equation (PDE)**, which involves unknown multivariable functions and their partial derivatives. For example, a PDE for the function $u(t_1, \ldots, t_n)$ is an equation of the form

$$F\left(t_1, \ldots, t_n, u, \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_n}, \frac{\partial^2 u}{\partial t_1 \partial t_1}, \ldots, \frac{\partial^2 u}{\partial t_1 \partial t_n}, \ldots\right) = 0. \tag{2}$$

The goals from Equations (1) and (2) are to find $y = y(t)$ and $u = u(t_1, t_2, \ldots, t_n)$, respectively.

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Remark 1.

In applications, the functions usually represent **physical quantities**, the derivatives represent their **rates of change**, and the differential equation defines a **relationship between the two**.

- Because such relations are extremely common, **differential equations play a prominent role in many disciplines** including engineering, physics, computer science, economics, biology, etc.

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An ODE, say

$$y' = f(t, y), \tag{3}$$

by itself **does not determine a unique solution function**, because the equation merely specifies the slopes of the solution $y'(t)$ at each point, but not the actual solution value $y(t)$ at any point.

To single out a **particular solution**, we must specify the value, usually denoted by y_0, of the solution function at some point, usually denoted by t_0. Thus, part of the given problem data is the requirement that

$$y(t_0) = y_0. \tag{4}$$

This additional requirement determines a unique solution to the ODE, provided that f is continuously differentiable. Here, y_0 is called the **initial value**, and (4) is called the **initial condition**. We refer to Equations (3) and (4) by the **initial value problem (IVP)**.

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Remark 2.

In common real-life situations, the IVP that models the problem is **TOO COMPLICATED** to solve exactly; therefore, we use **NUMERICAL METHODS** for approximating the solution of the original problem. This is the approach that is most commonly taken because the approximation methods **give more accurate results** and **realistic error information**.

One-Step Methods

Divide the domain of the solution into $N \in \mathbb{Z}^+$ subintervals using $(N+1)$ equally-spaced nodes $\{t_i\}_{i=0}^N$ with a common difference (step-size) h, and denote $y(t_i)$ by y_i. As the name suggests, **one-step methods compute a future prediction y_{i+1}, based only on information at the single point y_i and no other previous information**.

- All one-step methods can be expressed in the general form

$$y_{i+1} = y_i + \phi\, h, \tag{5}$$

where ϕ is called the *slope (increment function)*. The only difference between the one-step methods being the manner in which the slope is estimated.

Multi-Step Methods

- Multi-step approaches use information from **several previous points** as the basis for extrapolating to a new value. In particular, they use $k+1$ previously computed solution values $y_i, y_{i-1}, \ldots, y_{i-k}$ to compute the next solution value y_{i+1}.
  - This class of methods is generally **more accurate** than one-step methods.

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Euler's Method

Euler's method is the most basic numerical method for solving IVPs that belongs to the class of **one-step methods**. The method is named after the Swiss mathematician Leonhard Euler (1707–1783). Consider the IVP,

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha. \tag{6}$$

We subdivide the interval $[a, b]$ into N equal subintervals using the $(N + 1)$ mesh points $t_i = a + i\,h, i = 0, \ldots, N$, where $h = (b - a)/N$ is the step-size. Assuming that $y \in C^2[a, b]$ and using Taylor's Theorem to expand $y(t)$ about $t = t_0$ yields

$$y(t) = y_0 + (t - t_0)y_0' + \frac{(t - t_0)^2}{2!} y''(c), \tag{7}$$

where c lies between t_0 and t. At $t = t_1$, Equation (7) becomes

$$\begin{aligned} y_1 &= y_0 + (t_1 - t_0)y_0' + \frac{(t_1 - t_0)^2}{2!} y''(c), \\ &= y_0 + h\,f(t_0, y_0) + \frac{h^2}{2} y''(c). \end{aligned}$$

Euler's Method

If the step-size h is chosen small enough, then we may neglect the second-order term (involving h^2) and get the following difference equation:

$$y_1 \approx y_0 + h\, f(t_0, y_0),$$

The process is then repeated and generates a sequence of points that approximates the solution curve $y(t)$. The **general step for Euler's method** is

$$y_0 = \alpha, \tag{8a}$$

$$y_{i+1} \approx y_i + h\, f(t_i, y_i), \quad i = 0, \ldots, N-1. \tag{8b}$$

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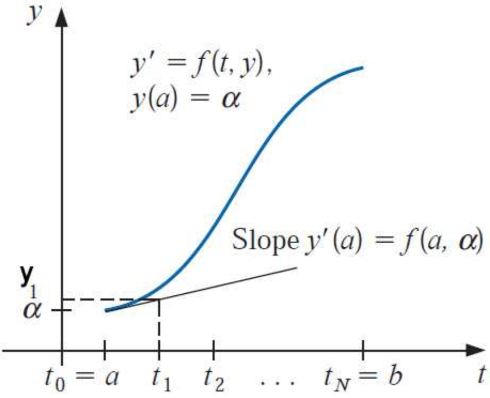
2 Euler's Method

- Graphical Derivation of Euler's Method
- Error analysis

Graphical Derivation of Euler's Method

Euler's method uses the ODE to evaluate the slope of the tangent at $(t_0, y(t_0)) = (a, \alpha)$. It then steps along the tangent to the point (t_1, y_1). Since,

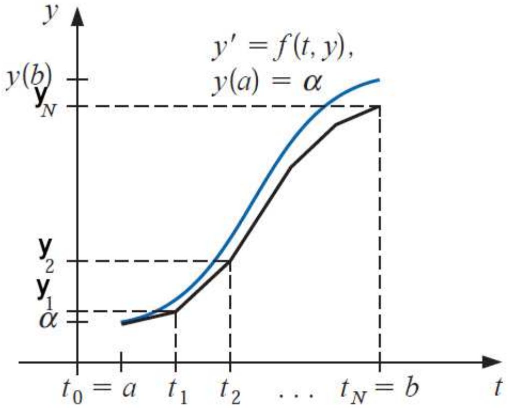
$$y_0' = \frac{y_1 - y_0}{h} \Rightarrow y_1 = y_0 + hy_0' = y_0 + hf(t_0, y_0).$$



Graphical Derivation of Euler's Method

The process is then repeated and generates a sequence of points that approximates the solution curve $y(t)$ such that

$$y_{i+1} = y_i + hf(t_i, y_i), \quad i = 0, \ldots, N-1.$$



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Error analysis

The **truncation errors** are composed of two parts.

1. The first is a **local truncation error** T_l that results from an application of the method in question over a single step.
2. The second is a propagated truncation error, called the **global truncation error** T_g, results from the total approximations produced during the previous steps.

When obtaining Euler's method, the neglected term for each step was $\frac{h^2}{2}y''(c_i)$. This neglected term represents the local truncation error T_l, and it measures the accuracy of the method at a specific step, assuming perfect knowledge of the true solution at the initial time step. Often we write,

$$T_l = \frac{h^2}{2}y''(c_i) = O(h^2), \text{ as } h \to 0, \quad (9)$$

to infer that **the local truncation error is of $O(h^2)$, as $h \to 0$**.

Error analysis

At the end of the interval $[a, b]$, after N steps have been made, the accumulated error (global truncation error T_g) would be

$$T_g = \sum_{i=1}^{N} \frac{h^2}{2} y''(c_i) = \frac{Nh^2}{2} y''(c) = \frac{(b-a)}{2} h\, y''(c),$$

$$\Rightarrow \boxed{T_g = O(h), \text{ as } h \to 0}, \tag{10}$$

where $y''(c)$ is the mean value for the discrete points $\{y''(c_i)\}_{i=1}^{N}$, for some $c \in (a, b)$. **Hence the global truncation error is of $O(h)$, as $h \to 0$.**

Euler's Method

Example 1

Use Euler's method with step-size $h = 0.5$ to approximate the solution of the IVP,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$.

Solution 1

For this problem, $f(t, y) = y - t^2 + 1$, *so* $y_0 = 0.5$,

$$\begin{aligned} y_1 &= y_0 + h(y_0 - t_0^2 + 1) = 0.5 + 0.5(0.5 - (0)^2 + 1) = 1.25, \\ y_2 &= y_1 + h(y_1 - t_1^2 + 1) = 1.25 + 0.5(1.25 - (0.5)^2 + 1) = 2.25, \\ y_3 &= y_2 + h(y_2 - t_2^2 + 1) = 2.25 + 0.5(2.25 - (1)^2 + 1) = 3.375, \\ y_4 &= y_3 + h(y_3 - t_3^2 + 1) = 3.375 + 0.5(3.375 - (1.5)^2 + 1) = 4.4375. \end{aligned}$$

Hence, $y(2) \approx y_4 = 4.4375$.